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Appendix A
General Information

A.1 Useful Data

A.1.1 Conversion Factors

Acoustics uses the metric system exclusively, but unfortunately American engineering still uses the English system (which the English have abandoned-except vehicle speeds and distances in the UK). As a result unit conversions are always necessary.

1 lb_f = 4.448 newtons = 444,800 dynes

1 lb_m = 0.4536 kilograms = 453.6 grams $= 0.0311\dfrac{lb_f - \sec^2}{ft}$

1 ft = 0.3048 meters = 30.48 centimeters

$1\dfrac{ft - lb_f}{\sec} = 1.356$ Watts = 1.356 newton-meter/sec

$1 lb_f = 32.2\dfrac{lb_m - ft}{\sec^2}$

1 mph =1.467 ft/sec = 0.447 m/sec = 0.869 knots

1 ft^3=28.33 liters=1728 in^3=0.02832 m^3

A.1.2 Reference Quantities

A.1.2.1 Speed of Sound in Air

For Celsius the speed is in meters/second, while for Fahrenheit it is in feet/second.

$$c_0 = 20.05\sqrt{(273.2 + {}^{\circ}C)}$$

$$c_0 = 49.03\sqrt{\left(459.7 + {}^{\circ}F\right)}$$

$${}^{\circ}F = \frac{9\,{}^{\circ}C}{5} - 32$$

$${}^{\circ}R = 459.67 + {}^{\circ}F$$

$${}^{\circ}K = 273.15 + {}^{\circ}C$$

For a standard temperature of 72°F the speed is 1130.5 ft/sec, or 344.4 m/sec.

A.1.2.2 Specific Acoustic Impedance of Air

This quantity represents the characteristics of the source surroundings and so is implicit in all acoustic equations. It is the ratio of the force at a point in a free progressive wave to the response (velocity) at the same point. By its nature it is resistive.

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$$Z_0 = \frac{p}{u} = \rho_0 c_0 \quad \text{N-sec/m}^3 \text{(mks Rayls)}$$

For 72°F and normal atmospheric conditions, Z_0=406 Rayls. Since English units are used here, it is necessary to convert this value.

$$Z_0 = \rho_0 c_0 = 406 \frac{N-\sec}{m^3} * \frac{1 lb_f}{4.448 N} * \frac{(1m)^3}{(3.281 ft)^3} = 2.584 \frac{lb_f - \sec}{ft^3} = 83.2 \frac{lb_m}{ft^2 - \sec}$$

The latter value is useful for known mass flow rates.

A.1.2.3 Density of Air at 70° F

$$\rho_0 = 0.0750 \frac{lb_m}{ft^3} = 0.00233 \frac{lb_f - \sec^2}{ft^4}$$

A.1.2.4 Viscosity of Air at 70° F

Dynamic $\mu = 1.22x10^{-5} \frac{lb_m}{ft - \sec} = 3.79x10^{-7} \frac{lb_f - \sec}{ft^2}$

Kinematic $\nu = \frac{\mu}{\rho} = 1.63x10^{-4} \frac{ft^2}{\sec}$

A.1.2.5 Gas Constant of Air

$$\gamma = 1.4, \ \text{R} = 53.3 \frac{lb_f - ft}{lb_m - {}^\circ R}$$

A.1.2.6 Two Important Constants

π=3.1415926535……
e=2.7182818284……

A.1.3 Level References

Sound Power $W_R = 10^{-12}$ Watts

Sound Pressure $\text{P}_\text{R} = 2\text{x}10^{-5}$ N/m^2 P_R=2.90x10^{-9} = lb$_f$/in^2

Intensity $\text{I}_\text{R} = 10^{-12}$ Watts/m^2

A.1.4 Levels

Sound Power Level $L_W = 10 \log_{10}\left(\frac{W}{W_R}\right)$

Sound Pressure Level $L_p = 10 \log_{10}\left(\frac{p^2}{p_R^2}\right) = 20 \log_{10}\left(\frac{p_{rms}}{p_R}\right)$

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A.2 Units and Dimensions

A.2.1 Dimensional Variables

Much of the analysis of sound problems depends on the correct determination of the dimensions of variables. Engineering units are susceptible to large errors, particularly conversion of mass to force. It is always necessary to make sure the dimensions of each term of an equation are the same. An example of some equations that are used elsewhere in this monograph is shown in Eqs. A.1. The exponent in the expression of *velocity potential* must be dimensionless, so each of the component terms must also be dimensionless. The pressure and velocity are related to the velocity potential {A.3.1}; the dimensions are indicated to the right of

$$\begin{aligned} &\phi = Ae^{i(\omega t - kr)}, A = ? \\ &p = \rho_0 \frac{\partial \phi}{\partial t}, \frac{F}{L^2} = \frac{MA}{L^3 T}, A = \frac{FLT}{M}\frac{ML}{FT^2} = \frac{L^2}{T} \\ &u = -\frac{\partial \phi}{\partial r}, \frac{L}{T} = \frac{A}{L}, A = \frac{L^2}{T} \end{aligned} \tag{A.1}$$

the expression. Solving for **A** yields differing results, until the conversion from mass units to force units is done. Checking dimensional consistency is the best way to find missing values. The number **32.2** used to convert mass to force is large in the engineering system, which is one reason why the metric system has been introduced. Unfortunately, acoustics, based on the much more logical metric system, is embedded in a sea of engineering units in this country.

Since dimensional consistency is important, the dimensions of the variables used in this monograph are given below. These are useful to check the dimensions of any equation.

Time	T
Length	L
Force	F
Mass	M
Frequency	1/T
Wave number	L^{-1}
Speed	L/T
Velocity Potential	L^2/T
Pressure	F/L^2
Dynamic Viscosity	FT/L^2
Kinematic Viscosity	L^2/T
Impedance	FT/L^3
Density	$FT^2/L^4=M/L^3$
Volumetric Flow Rate	L^3/T
Force Moment	FL
Intensity	F/LT
Power	$FL/T=ML^2/T^3$

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A.2.2 Dimensionless Variables

Nature knows nothing of the dimension system we use; it is concerned only with the ratios of things. Dimensional similarity refers to the ratio of dimensions preserved when an object's size is changed. Dynamic similarity refers to the ratio of forces preserved when the forces are changed. Dimensionless variables are an extremely useful way to take changes into account. If the dimensionless variables do not change when the dimensional variables change, similarity is achieved and tests in one situation can be extrapolated to another. This is quite important in sound generation. The major dimensionless variables associated with fluid mechanics are derived in {D.2}. They imply a choice of two *characteristic* dimensional variables, a speed **U** and a length **L** that are used to create the dimensionless variables. They relate to both the forces and geometry involved. The art is in choosing these two important variables. All the examples in this monograph use this art to show how sound problems can be approached in new situations.

A.2.2.1 Strouhal number

$$St = \frac{fL}{U}$$

The ratio of unsteady inertial forces to steady inertial forces. The number is named in honor of Vincenz Strouhal (1850-1902) who first deduced the relationship between the vortex shedding frequency around a cylinder, the cylinder diameter **L**, and the speed of the flow over it **U**. The number was found to be virtually constant from a Reynolds number of 500 to 200000. This number permits relationships to be developed between different sizes, speeds, and time variables rates. This number is developed from the mass continuity equation {D.2}. It can be considered to be a *hydrodynamic Strouhal number* since the characteristic speed is that of the fluid. The numerical value of the Strouhal number would better match the other variables if the frequency were expressed in radians/second {A.2.2.2}.

A.2.2.2 Helmholtz number

$$He = kL = \frac{\omega L}{c_0} = \frac{2\pi fL}{c_0} = \frac{2\pi L}{\lambda} = 2\pi St_a$$

The characteristic length **L** is expressed in sound wavelengths. The characteristic speed is that of sound. The variable **k** is called the *wave number* despite the fact that its dimensions are L^{-1}. The wave number is integral to the mathematical development of sound sources. The attribution to Hermann Helmholtz (1821-1894) is that the characteristic length **L** was the radius of a tube and the specific value of the number referred to specific frequencies of a tube resonance. Note the similarity of the Strouhal number to this number; the difference being the characteristic speed. Since the Strouhal number is commonly used, the relationship above is also expressed as an *acoustical Strouhal number* where the characteristic speed is that of sound. When used in this form, the subscript **a** will be used.

A.2.2.3 Mach number

$$Ma = \frac{U}{c_0}$$

The ratio of the steady speed to the speed of sound. The number is named in honor of Ernst Mach (1838-1916) who first studied (among other things) supersonic motion and the shock waves generated. He developed a method to see the otherwise invisible shock structure. This number permits identification of the separation point between incompressible and compressible flow. This variable is developed from the momentum equation.

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A.2.2.4 Reynolds number

$\mathrm{Re} = \frac{UL}{\upsilon}$ The ratio of the steady inertial forces to the steady viscous forces. The number is named in honor of Osborne Reynolds (1842-1912), an engineer who did pioneering studies on the transition of laminar to turbulent flow in pipes. This number permits relationships to be developed between different sizes, speeds and fluids. This variable is developed from the momentum equation.

A.2.2.5 Rossby number

$\mathrm{R}o = \frac{U}{f_0 L}$ The ratio of linear velocity to tangential velocity for swirl flows. The frequency is characteristic of the rotation rate of the flow. The number is named in honor of Carl-Gustav Rossby (1898-1957), a meteorologist who first described the large scale motions of the atmosphere in terms of fluid mechanics. He described the jet stream, and his number was first used to describe the motion associated with the coriolis force in the atmosphere. This variable is developed elsewhere from the equations of motion in curvilinear coordinates.

A.2.2.6 Frequency

$\omega t = 2\pi\, ft = \frac{2\pi c_0 t}{\lambda} = kc_0 t$ It can be considered a dimensionless frequency or the ratio of the distance sound travels in time **t** to the sound wavelength.

A.2.2.7 Force

$\widehat{F} = \frac{F}{\rho_0 U^2 L^2}$ The ratio of the actual dynamic force to the steady momentum.

A.2.2.8 Force Moment

$\hat{\tau} = \frac{\tau}{\rho_0 U^2 L^3}$ The ratio of the dynamic stress moments to the steady stress moments.

A.2.2.9 Volumetric Flow Rate

$\widehat{Q} = \frac{Q}{UL^2}$ The ratio of the dynamic volumetric flow rate to the steady volumetric flow rate.

A.2.2.10 Power

$\widehat{W} = \frac{W}{\rho_0 U^3 L^2}$ The ratio of the dynamic (sound) power to the steady power.

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A.3 Physical Variables Derived from Velocity Potential

A.3.1 In The Time Domain

The velocity potential in spherical coordinates is **ϕ(r, θ, ψ, t).** The following physical acoustical variables can be derived shown in Eqs. A.2.. The dimensions of each variable are also shown.

$$\begin{aligned}
&\phi(r,\theta,\psi,t), \frac{L^2}{T} \\
&p(r,\theta,\psi,t) = \rho_0 \frac{\partial\phi}{\partial t}, \frac{F}{L^2} \\
&u_r\left(r,\theta,\psi,t\right) = -\frac{\partial\phi}{\partial r}, \frac{L}{T} \\
&u_\theta\left(r,\theta,\psi,t\right) = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{L}{T} \\
&u_\psi\left(r,\theta,\psi,t\right) = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}, \frac{L}{T} \\
&s(r,\theta,\psi,t) = \frac{1}{c_0^2}\frac{\partial\phi}{\partial t} = \frac{\rho-\rho_0}{\rho_0} \\
&T(r,\theta,\psi,t) = \frac{1}{R}\left(\frac{\gamma-1}{\gamma}\right)\frac{\partial\phi}{\partial t} \\
&I_r(r,\theta,\psi) = p \bullet u_r^* = -\rho_0 \frac{\partial\phi}{\partial t} \bullet \frac{\partial\phi^*}{\partial r} \\
&W = \int_0^{2\pi}\int_0^{\pi} I_r(r,\theta,\psi) r^2 \sin\theta d\theta d\psi
\end{aligned} \qquad \text{(A.2)}$$

The next to last term is the density fluctuation and is called *condensation*, note that it is dimensionless. It is expressed in lower case to distinguish it from the Strouhal number. Note that pressure, density and temperature are all in phase as might be expected with fluid compression.

A.3.2 In the Frequency Domain

The velocity potential in spherical coordinates is **Φ (r, θ, ψ, ω)**. It is related to the time domain through the relationship

$$\Im[\varphi] = \Phi\left(r,\theta,\psi,\omega\right) = \int_{-\infty}^{\infty} \varphi\left(r,\theta,\psi,\tau\right) e^{-i\omega\tau} d\tau$$

In a similar manner, the following physical variables can be derived from it.

$$p(r,\theta,\psi,\omega) = \rho_0 \Im\left[\frac{\partial \varphi}{\partial t}\right] = i\omega\rho_0\Phi$$
$$u_r\left(r,\theta,\psi,\omega\right) = -\frac{\partial \Phi}{\partial r}$$
$$u_\theta\left(r,\theta,\psi,\omega\right) = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}$$
$$u_\psi\left(r,\theta,\psi,\omega\right) = -\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \psi}$$
$$s(r,\theta,\psi,\omega) = \frac{ik\Phi}{c_0}$$
$$T(r,\theta,\psi,\omega) = \frac{i\omega}{R}\left(\frac{\gamma-1}{\gamma}\right)\Phi$$

A.4 Fourier Series

A.4.1 Real Fourier Series

Simple harmonic motion is restricted to one frequency. Complex harmonic (periodic) motion contains several frequencies. Examples are: square and triangular waves. To handle them, one must expand the time history by using the Fourier series expansion of that function. The basic equations are:

$$f\left(x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right)$$
$$a_n = \frac{1}{T}\int_{-T}^{T} f\left(x\right)\cos\left(\frac{n\pi x}{T}\right)dx$$
$$b_n = \frac{1}{T}\int_{-T}^{T} f\left(x\right)\sin\left(\frac{n\pi x}{T}\right)dx$$

$\mathbf{a_0}$ is the steady value. It is generally set to zero for acoustical applications. **T** is the half period.

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Example

A square wave of half-amplitude **A** has a period **2T** of two around a value of 0. Thus the variable **T** is 1, and $\mathbf{a_0}$ is zero. **f(x)=-A** from –1 to 0 and **f(x)=A** from 0 to 1. The integrals are shown in the next set of equations.

$$a_n = -\int_{-1}^{0} A\cos\left(n\pi x\right)dx + \int_{0}^{1} A\cos(n\pi x)dx = 0$$

$$b_n = -\int_{-1}^{0} A\sin\left(n\pi x\right)dx + \int_{0}^{1} A\sin(n\pi x)dx = \frac{A}{n\pi}\left[1-\cos\left(n\pi\right)\right]$$

$$b_n = \frac{4A}{n\pi}$$

$$b_n = 0$$

Odd values of n have finite values.

$$f(x) = 4A \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi}$$

A square wave was constructed of a sinusoid of period 2; the summation of its odd harmonics are shown in the figures below. There is always an overshoot at the transition with a finite number of terms. Albert Michelson (1852-1931) noticed this but attributed it to error. J. Willard Gibbs showed that it was an artifact of the Fourier series. It is based on approximating a discontinuous function with a finite series of continuous functions. It is now call the Gibbs phenomenon. Figure A-1 shows a 9 term fit to a square wave. Figure A-2 shows the Gibbs phenomenon more clearly with 105 terms.

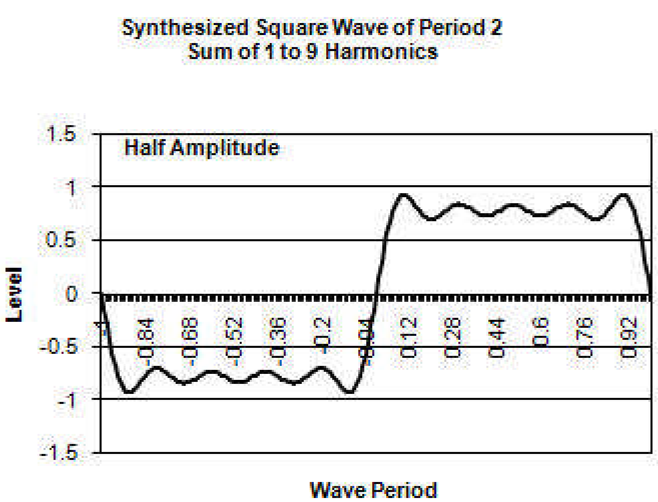

Fig. A-1. A 9 term fit of Fourier series to a square wave.

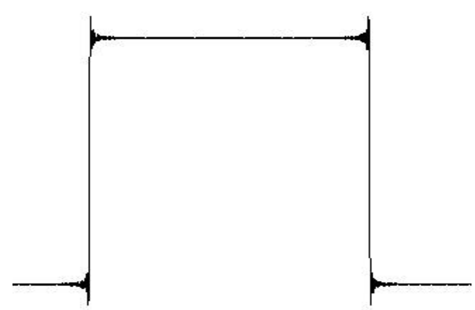

Figure A-2. A 105 term fit of Fourier series to a square wave.

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A.4.2 Fourier Integrals

For functions that are more complex, such as random sound or transient motions, it is necessary to work in the frequency domain. That is the purpose of the Fourier Integral. There are two equations that act as transforms from one form to another, one in the time domain and the other in the frequency domain. They are called *Fourier Transform pairs*. In complex form they are:

$$F(\omega) = \Im\left[f(t)\right] = \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}d\tau$$

$$f(t) = \Im\left[F(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

The second form of the equations is shorthand to describe that a Fourier transform has occurred without writing out the equation itself. The value of this transform can be seen if the simple mass, spring, oscillator equation is used; it is

$$M\ddot{x}(t) + Kx(t) = f(t)$$

Each dot represents a differentiation of the displacement **x** and **f** represents an arbitrary force. If the force is considered harmonic, the solution is straightforward, but since it is arbitrary, it is not. Using the transform, the equation form reduces to one in which frequency spectrum replaces the time.

$$M\Im\left[\ddot{x}(t)\right] + K\Im\left[x(t)\right] = \Im\left[f(t)\right]$$

We are now dealing with the frequency spectrum of the force. If the derivatives of the displacement can be determined, the problem can be solved. See below

The real form of the integral is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A(\omega)\cos\omega t + B(\omega)\sin\omega t\right]d\omega$$

$$A(\omega) = \int_{-\infty}^{\infty} f(\tau)\cos\omega\tau d\tau$$

$$B(\omega) = \int_{-\infty}^{\infty} f(\tau)\sin\omega\tau d\tau$$

A.5 Complex Fourier Transforms

A.5.1 Stationary

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty} F(\omega)e^{i\omega t}d\omega = \frac{1}{\pi}\int_{0}^{\infty} F(\omega)^{*}e^{-i\omega t}d\omega$$

$$\Im\left[f(t)\right] = F(\omega) = \left|F(\omega)\right|e^{i\theta} = \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}d\tau$$

A.5.2 Transient

$$f(t) = \frac{2}{\pi}\int_0^\infty F(\omega)e^{i\omega t}\,d\omega$$

$$F(\omega) = \int_0^\infty f(\tau)e^{-i\omega\tau}\,d\tau$$

A.6 Cosine Fourier Transforms

A.6.1 Stationary

$$f(t) = \frac{1}{\pi}\int_0^\infty F_c(\omega)\cos\omega t\,d\omega$$

$$F_c(\omega) = |F_c(\omega)|\cos\theta = \int_{-\infty}^\infty f(\tau)\cos\omega\tau\,d\tau$$

A.6.2 Transient

$$f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega t\,d\omega$$

$$F_c(\omega) = \int_0^\infty f(\tau)\cos\omega\tau\,d\tau$$

A.7 Sine Fourier Transforms

A.7.1 Stationary

$$f(t) = \frac{1}{\pi}\int_0^\infty F_s(\omega)\sin\omega t\,d\omega$$

$$F_s(\omega) = |F_s(\omega)|\sin\theta = \int_{-\infty}^\infty f(\tau)\sin\omega\tau\,d\tau$$

A.7.2 Transient

$$f(t) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega t\,d\omega$$

$$F_s(\omega) = \int_0^\infty f(\tau)\sin\omega\tau\,d\tau$$

A.8 Transform Relationships

$$F(\omega) = F_c(\omega) - iF_s(\omega)$$
$$|F(\omega)|^2 = F(\omega)F(\omega)^* = F_c(\omega)^2 + F_s(\omega)^2$$
$$Tan\theta = F_s(\omega) / F_c(\omega)$$
$$\Im\left[\frac{df(t)}{dt}\right] = -f(\beta) + i\omega F(\omega)$$
$$\Im\left[\frac{d^2 f(t)}{dt^2}\right] = -\frac{df(\beta)}{dt} - i\omega f(\beta) - \omega^2 F(\omega)$$
$$f(\beta) = 0, \frac{df(\beta)}{dt} = 0, stationary$$
$$f(\beta) = f(0), \frac{df(\beta)}{dt} = \frac{df(0)}{dt}, transient$$
$$\Im\left[f(t+\tau)\right] = e^{-i\omega\tau} F(\omega)$$

The equations above are limited to positive frequencies, since it is not possible to distinguish positive from negative. The factor of 2 appears in the cosine and sine integrands to make them even functions so that the time function is real. For the transient cases, the integrals have been restricted to positive times resulting in another factor of 2.

A.8.1. Square Pulse

A pulse of amplitude **H** occurs for a time **T**.

$$F(\omega) = H\int_0^T e^{-i\omega\tau} d\tau = \frac{iH}{\omega}\left[e^{-i\omega T} - 1\right]$$
$$F(\omega) * F^*(\omega) = \frac{2H^2}{\omega^2}\left[1 - \cos\omega T\right] = \frac{2H^2}{\omega_1^2}\left[\frac{1 - \cos\pi\alpha}{\alpha^2}\right]$$

The equations below define a ratio of frequencies to the fundamental periodicity of the pulse.

$$\omega_1 T = \pi$$
$$\alpha = \omega / \omega_1$$

Figure A-3 show the relative levels as a function of frequency in terms of the ratio **α.** Note that minima occur at even multiples of the base period.

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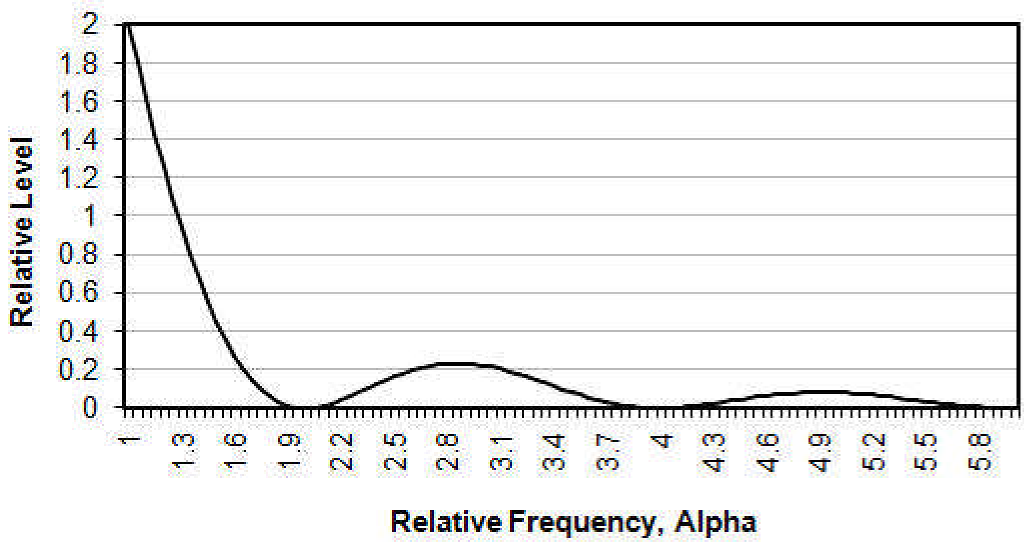

Fig. A-3. The frequency spectrum of a square pulse.