Appendix B
Geometry and Complex Operations

B.1 Spherical Geometry and the Wave Equation

The equations of fluid motion are embodied in the equations of mass, momentum and thermodynamics. One result of linearization is the wave equation shown below in spherical coordinates. This equation suggests that wave motion must be embedded in the more complex motions of fluids and care must be taken to define the limitations of the equation and the approximations needed for its validity. This is done in Appendix D. The equation describes only wave motion and has no indication of the source of the motion. Thus it must be applied away from any source regions. Some texts include a source term, but it will be shown that much can be learned without it. All fluids have viscosity, but the wave equation excludes it.

The wave equation in spherical coordinates is

$$\square^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}-\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2}=0 \tag{B.1}$$

where **r** is the radial coordinate, **θ** and **ψ** are polar coordinates. The relation to Cartesian coordinates is shown in the equations below. The element of area on the surrounding spherical surface is shown in Figure B-1. To integrate over the sphere, **θ** must vary from 0 to π, and **ψ** must vary from 0 to 2π.

Eq. B.1 is best expressed in terms of a *velocity potential* {A.6.1} from which the physical variables can be readily derived. These relationships are given below.

$$\begin{aligned}
p &= \rho_0\frac{\partial\phi}{\partial t}\\
u_r &= -\frac{\partial\phi}{\partial r}, u_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, u_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\\
s &= \frac{1}{c_0^2}\frac{\partial\phi}{\partial t}\\
x &= r\sin\theta\cos\psi\\
y &= r\sin\theta\sin\psi\\
z &= r\cos\theta\\
dA &= r^2\sin\theta\, d\theta\, d\psi
\end{aligned} \tag{B.2}$$

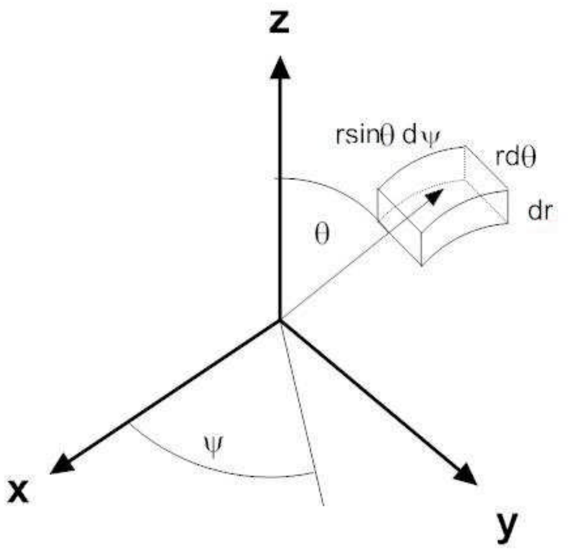


Fig. B-1. Spherical coordinates.

B.2 Complex Notation

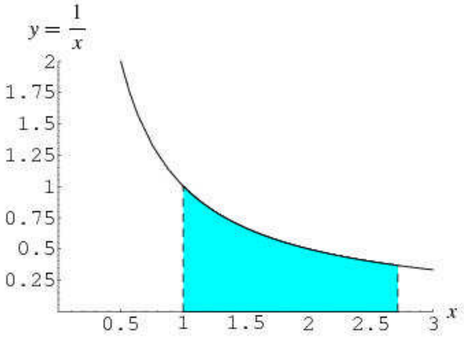
B.2.1 The Constant e

The constant **e** is used in all areas of mathematics and is very useful in the complex notation used for sound generation. An early form is given below.

$$e^{i\pi} + 1 = 0$$

Euler's (Napier's) Identity: "The most beautiful theorem in math"
It ties the two most used irrational numbers to the imaginary operator and the integers all in one relationship.

But what is the meaning of e? There are as many ways of defining it as there are ways to use it. One is to define it as the area under the hyperbolic curve shown in Figure B.2.



$$\int_1^e \frac{dx}{x} = \ln e = 1$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\sum_{k=0}^{n-1} e^{\frac{2\pi ik}{n}} = 0, n > 1$$

Fig. B-2 The value of e as an integral limit.

The integral for it is given in the first equation to the right. It is the value of the upper limit that makes the area under the curve equal to one. It is the base of the natural (Naperian) logarithms. It can be defined as an infinite series as shown in the second equation. The third expression is a generalization of the identity shown at the section beginning that covers all the zeroes. The number (e=2.7182818284….) is referred to as Euler's number since it was the Swiss mathematician Leonard Euler (1707-1783) that made use of it. It is also referred to as Napier's number since John Napier used it as the base of logarithms. Complex notation makes extensive use of this number.

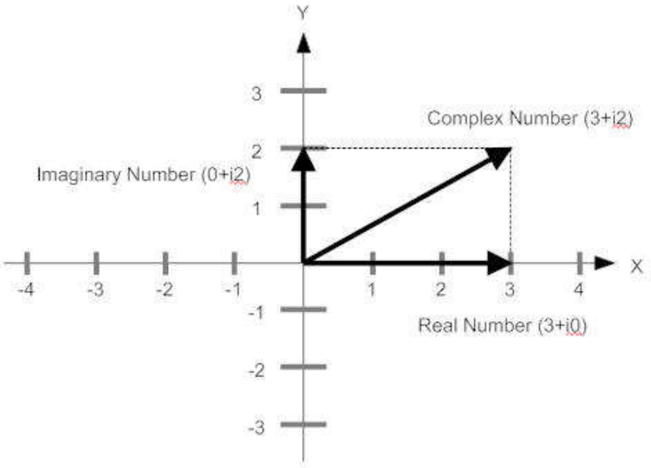


Fig. B-3 Complex coordinates.

B.2.2 Simple Harmonic Motion

Simple harmonic motion is best described using complex notation. Figure B-3 shows what is called the *real* and *imaginary* axes. Any point on the plane must be described by two numbers. The complex number, 3+i2, is the end of the arrow in the figure. The letter **i** (**j** is used in some engineering documents) is added to indicate the imaginary axis.

It is important to note that $i = \sqrt{-1}$ has two roots, and it was first used by Gerolomo Cardano (1501-1575), who first described typhoid fever. One must stand in awe at the cleverness of the first person to express complex geometry in terms of an irrational number raised to the power of an imaginary number, e.g., $e^{i\pi}$.

Typical descriptors in polar coordinates are shown in Figure B-4. The arrow can be described by the linear dimensions or the radius and angle. Putting these two figures together we get the following relationships.

$$R = a + ib = R\left(\cos\theta + i\sin\theta\right) = \sqrt{a^2 + b^2}\, e^{i\tan^{-1}\left(\frac{b}{a}\right)} = \mathrm{R}e^{i\theta}$$

$$\left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta = e^{in\theta} \qquad \text{(B.3)}$$

The latter formula comes from a theorem by Abraham de Moivre (1667-1754) through use of infinite series expansions.

The main advantage of using the exponential form is *ease of manipulation*. Consider the addition of two complex numbers, x=a+ib and y=c+id. The rules of algebra, give the simple result x+y=(a+c)+i(b+d).

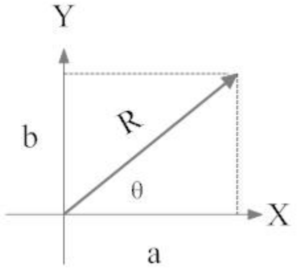


Fig. B-4 Polar variables.

Consider multiplication of two complex numbers. When doing so, $\mathbf{i}^2$ is encountered. Other manipulations of the imaginary number are often needed. Table B-1 below shows a table of results for common operations with that number. The equation for one calculation is shown on the right of the table.

i^0	i^1	i^2	i^3	i^4	i^{-1}	i^{-2}	i^{-3}	i^{-4}
1	i	-1	-i	1	-i	-1	i	1

$$i = e^{\frac{i\pi}{2}}$$

$$i^2 = e^{i\pi} = -1$$

Table B-1. Manipulating the square root of minus one.

Eqs B.3 can be used to convert each variable to polar form as shown on the right. Carrying out the multiplication in polar form gives a simpler result. The arithmetic approach yields xy=(ac-bd)+i(bc+ad). This form is difficult to interpret when dealing with frequencies and phases, so polar notation is used throughout this monograph.

There is one other complex number operation that is useful in sound analyses: the *complex conjugate* and is often denoted by an exponent *; the sign of the imaginary *operator* is reversed. The sound pressure can be described in complex notation, but to calculate mean square pressure, the equation must be reduced to its real component, which is easy with complex notation as shown on the right.

$$x = a + ib = Ae^{i\theta}$$

$$y = c + id = Be^{i\phi}$$

$$x + y = Ae^{i\theta} + Be^{i\phi}$$

$$xy = ABe^{i(\theta+\phi)}$$

$$p = a + ib = P\,\mathrm{e}^{i\theta}$$

$$p^* = a - ib = P\,\mathrm{e}^{-i\theta}$$

$$p \bullet p^* = P^2 = a^2 + b^2$$

B.3 The Power of Complex Notation

To develop solutions for dynamics problems it is often necessary to start from first principles. Unfortunately, there is very little guidance on how this is done. In this section, the classic vibrating mass-spring problem is solved using first principles (almost), and then more modern methods.

B.3.1 The Original Method

There is a mass suspended from a spring and damper (Figure B-5) and the question is: What happens when the mass is disturbed? The addition of a damping mechanism is not needed for this example. The answer to this question can be found in textbooks [6].

The static equation and the equation of motion using Newton's law are given below. The subscript zero refers to the static position when no spring force is exerted (**x=x₀**). The subscript one refers to the static position with the weight attached.

$$F = Mg - K\left(x + x_1 - x_0\right)$$
$$F = M\frac{\partial^2 x}{\partial t^2} \qquad (B.4)$$

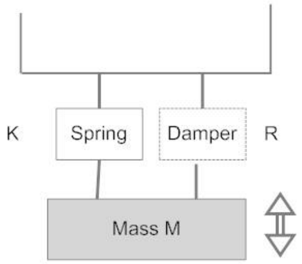


Fig. B-5. Mass-spring vibrator.

The solution for displacement will be developed by use of power series expansions, just as did Friedrich Bessel (1784-1846) to laboriously calculate by hand each of the terms of the cylindrical functions originated by Daniel Bernoulli (1700-1782). These functions are now known as Bessel functions. The equations are

$$\begin{aligned}
& x = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 ..... \\
& M\ddot{x} + Kx = Mg - K\left(x_1 - x_0\right) \\
& Mg - K\left(x_1 - x_0\right) = 0 \\
& \ddot{x} + \frac{K}{M}x = 0 \\
& a_2 = -\frac{Ka_0}{2M}, a_3 = -\frac{Ka_1}{6M}, a_4 = -\frac{Ka_2}{12M} = \left(\frac{K}{M}\right)^2 \frac{a_0}{24} \\
& x = a_0\left[1 - \left(\frac{K}{M}\right)\frac{t^2}{2!} + \left(\frac{K}{M}\right)^2 \frac{t^4}{4!} - .....\right] + a_1\left[t - \left(\frac{K}{M}\right)^2 \frac{t^3}{3!} + ...\right]
\end{aligned} \qquad (B.5)$$

The first equation is the series expansion of displacement **x** at time **t**, the second is the equation of motion where the double dot represents the second time derivative. Since the left side of the equation depends on time and the right does not, they must both be independent and equal to a constant which can be set to zero (free vibration), yielding the third and fourth equations. When the displacement **x** is differentiated twice and substituted, and the various time terms are equated to each other, we get relationships between the various constants as shown in the fifth equations. Note that each of the higher constants can be related to the first two constants. As a result, the last equation can be written as a series expansion of just two terms. We often forget, and take for granted, the huge base of knowledge created by these early investigators. They examined the two expansions and found that they relate to triangles, namely the sine and cosine functions (which themselves are infinite series expansions), so the form can be expressed as shown on the right.

$$x = a_0 \cos\left(\sqrt{\frac{K}{M}}t\right) + a_1\sqrt{\frac{M}{K}}\sin\left(\sqrt{\frac{K}{M}}t\right)$$

$$\omega = \sqrt{\frac{K}{M}}$$

Since the problem is dynamic, the square root function must be interpreted as a frequency (dimension $\mathbf{t^{-1}}$). This frequency is called the *natural frequency*, which is most *unnatural*. Since all real systems have damping it requires that the actual frequency always be less than the *natural* frequency.

B.3.2 The Complex Notation Method

The beauty of using complex notation, as is done throughout this monograph, is shown in Eqs. B.6. The displacement x is expressed in complex notation.

$$\begin{aligned} &x = Ae^{i\omega t}, \ddot{x} = -\omega^2 x \\ &-\omega^2 Mx + Kx = Mg - K\left(x_1 - x_0\right) \\ &\omega^2 = \frac{K}{M} \end{aligned} \qquad \text{(B.6)}$$

Since both the left and right sides of the second equation are independent, we get the third equation directly; much easier.

B.4 Simple Harmonic Motion

$$Re^{i\omega t} = R\cos\omega t + iR\sin\omega t$$

$$Re^{i(\omega t+\theta)} = R\cos(\omega t+\theta) + iR\sin(\omega t+\theta)$$

We can use complex notation to represent simple harmonic (single frequency) motion. If we think of the vector **R** as rotating in a circle in the complex plane, we can write the equations on the right where **ω** is the radian frequency, **2πf**. If the frequency is one cycle per second, the **R** vector rotates a complete circle once a second. This notation is very convenient for manipulating frequency information. The **θ** symbol is used to denote a phase angle which simply changes the time of axis crossing. The use of the imaginary axis in harmonic motion is simply a clean method of accounting for the instantaneous phase of the motion.

There are two methods of determining the "real" part of the motion (e.g., the mean square sound pressure) and they give *different* results. One method, shown in the first of Eqs. B.7 below, is to square the real part of the variable (it eliminates negative values) and then integrate over a period to get the desired result. Taking the square root yields root-mean-square (r.m.s.) values. The preferred method is to use complex notation as shown in the second of Eqs. B.7. The mathematical manipulation is simple, but the results are not the same as with the first method. The value of $\mathbf{p_2}$ must be interpreted as the *maximum amplitude* of the motion while p_1

must be interpreted as the *r.m.s. amplitude* (i.e., $\mathbf{p_1=0.707^* \, p_2}$ for simple harmonic motion). When dealing with random motion, it is not possible to define a peak amplitude at any particular frequency, but it is possible to define r.m.s. amplitudes, so the exponential form is preferred to the trigonometric form.

$$P_2 = p_2 \cos \omega t, P_2^2 = p_2^2 \int_0^{\frac{2\pi}{\omega}} \cos^2 \omega t dt = \frac{p_2^2}{2} \tag{B.7}$$
$$P_1 = p_1 e^{i\omega t}, P_1^2 = P_1 \bullet P_1^* = p_1 e^{i\omega t} \bullet p_1 e^{-i\omega t} = p_1^2$$

The constants in all equations for sound intensity and sound power in this monograph will be a factor of two larger than those shown in most textbooks that use trigonometric functions to derive the equations. Typical values will be 1/4 as opposed to 1/8.