Appendix C

Random Processes

C.1 Introduction

The specification and description of random processes is a broad field of science that overlaps many engineering disciplines. The basic concepts are described here, but it is not a detailed treatment.

A random process is one in which a precise prediction of an occurrence cannot be made, only the likelihood, or probability, of its occurrence can be described. Since all that is generally available to an engineer is a *time history* of a process, it is necessary to develop methods with which this history can be described usefully. All theory of random processes is concerned with *ensembles* and for our purposes they may be considered as an infinite array of time histories (requiring an infinite number of experiments). One description of a random process is given by *expectation value*s such as: $E\{x(t)\}$, $E\{x^2(t)\}$, $E\{x(t), x(t+\tau)\}$. These might be averages of the infinity of time histories recorded by some type of meter. If the first expected value noted above is zero, the process has a zero mean (typical of sound). The second term is the expected value of the next higher moment. When all the higher moments are known the process is completely described. The third term is the expected value of a process at a later time τ. In most engineering applications, only the first two moments are generally known. For sound sources of interest here, our concern is mostly with the second moment, typically mean square sound pressure.

Since most real world situations provide only one time history, the *Ergodic Hypothesis* must be invoked. For practical applications, the hypothesis requires that the process be stationary (the average of a time history over one time period is the same as the average over a different time period). This should apply to all moments of the sample, but the first and second moments are only of concern, so it applies only partially. As a result, in the description of a random process we are forced to assume that any time history is representative (typical) of any others that might be taken. This process runs into trouble with transients, particularly those which occur only once. At this point the theory diverges from practice, but that does not imply that no useful information can be gained.

C.2 Correlation or Phase

Phase relationships and correlations between sources are important. The technique of “wave cancellation” is well known: set two single frequency signals out of *phase* and they cancel – no sound (wherever that can be accomplished spatially). Consider that many sounds have a broad-band spectrum, typically random, for which it is not possible to define a phase. For that case, the words *correlation* or *covariance* are used. To demonstrate these concepts, consider a simple set of circumstances. There are two sound pressures from separate sources, $\mathbf{p_1}$ and $\mathbf{p_2}$, coming together at a particular location. What is the consequence of their interaction? The relevant relationships are shown in Eqs. C.1.

The symbol Δ is the ratio of the sum **p** to **p₁**. The bar represents the time average and $\mathbf{p_R}$ is the reference pressure for sound level. If $\mathbf{p_2}$ is exactly equal to $\mathbf{p_1}$ (fully correlated) then Δ = 6 dB. If $\mathbf{p_2}$ has the same magnitude as $\mathbf{p_1}$ but the time average of their product is zero (uncorrelated) then Δ = 3 dB. If $\mathbf{p_2}$ is exactly equal to $-\mathbf{p_1}$ (negatively correlated) then Δ = -∞ dB. Figure C-1 shows the complete range of correlation coefficients from full positive (1) to full negative (-1) for several relative magnitudes. The obvious observation is that the addition of fully correlated sources varies from 0 to 6 dB, depending on relative magnitudes, while for uncorrelated sources it varies from 0 to 3 dB. Less obvious is the fact that the addition of slightly negatively correlated source results in *no increase in level*, despite the *doubling* of the sound source; the required correlation is the relative magnitude divided by two.

$$L_p = 10\log_{10}\left[\frac{\overline{p^2}}{p_R^2}\right]$$
$$p = p_1 + p_2$$
$$\overline{p^2} = \overline{p_1^2} + \overline{p_2^2} + 2\overline{p_1 p_2} \qquad \text{(C.1)}$$
$$\Delta = 10\log_{10}\left[\frac{\left[1+\frac{\overline{p_2^2}}{\overline{p_1^2}}+2\frac{\overline{p_1 p_2}}{\overline{p_1^2}}\right]}{p_R^2}\right]$$

If we are concerned with single frequency sources then the correlation coefficients can be replaced by phase angles between the sources. In-phase (0°) relates to a coefficient of 1, quadrature (90°) relates to a coefficient of 0 and out-of-phase (180°) relates to a coefficient of -1. The phase angle equation is cosθ = relative magnitude divided by two. Representative phase angles to a whole degree are given in Table C-1 below.

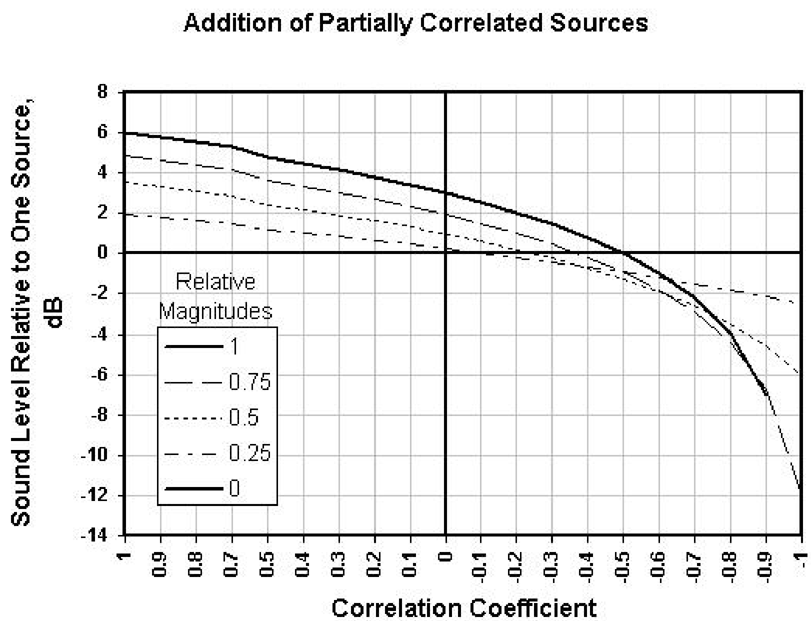

Fig. C-1. The addition of two partially correlated sources.

Relative Magnitude	1	0.75	0.5	0.25	0
Angle	120	112	105	97	90

Table C-1. The phase angle between two sources for no level increase.

Key Point: Phase relationships (or correlations) are important between sources. Since a sound field is spatial, the relationship is often purely local and not generally applicable.

C.3 Two Correlation Questions

With the full theory of sound generation [12], an integral over a distribution of sources is performed. It completely accounts for the phase relationships, or correlations, of the sources. Unfortunately, it cannot be solved in the general case. The approach in this monograph is to replace integrals with summations. Although appealing and simplifying, it leads to two fundamental questions.

C.3.1 What is the correlation between the various frequencies of one source?

There is no issue if the source is single frequency. If the source is periodic, but not sinusoidal (e.g., a square wave), Fourier analysis shows a harmonic structure. The level at each harmonic frequency has a defined phase relationship to the fundamental and is taken into account at a distance, since the waveform is preserved.

It is common practice to collect broad-band random sound data in one-third or one-octave bands. How are the bands to be summed to get the overall level? If the sound is caused by turbulent flows, the question becomes: What is the correlation between the frequencies associated with different eddy sizes at a particular location? It is generally assumed that such bands can be added incoherently, but it may not be valid always. Incoherent addition is more likely to be justified when the bandwidth is wide. One-third octave bands are added incoherently in this monograph.

C.3.2 What is the correlation between spatially separate sources?

Much of the modeling in this monograph is associated with breaking a distributed source into a finite number of individual uncorrelated sources and then summing them to get the overall level at a particular location. The interaction between two single frequency sources can vary from complete correlation to none. In several of the examples in this monograph, the second source is an image of the first, e.g. reflection from a surface. In this case, they are completely correlated and summing independent sources fails. The case of partial correlation, such as reflection from an absorptive surface is not addressed.

The interaction of two broad-band sources is more interesting. The sound from the trailing edge of an airfoil caused by turbulent boundary layer flow is one example. The various frequency bands are created by the turbulent eddies, and as noted above are added incoherently to determine overall. But what is the relationship between two eddies that pass the trailing edge simultaneously, but are laterally separated? There has to be a distance at which where they are not aware of each other and they can be treated as incoherent (independent) sources? It is highly likely that the distance depends on eddy size and therefore is a function of frequency. Unfortunately, this information is not available for many source sources and without this knowledge, the validity of modeling by summing independent sources is weak. The approach taken here is to model a sound source with a variable number of independent sources and compare the difference that occur. If data were available, the results could be compared with experiment to determine which distance is most correct.

C.4 Correlation and Covariance Equations

Correlations are used to relate two random variables: $\mathbf{x(r_1,t)}$ and $\mathbf{y(r_2,t)}$. The *mean value* for each variable can be expressed as

$$\eta_x(r_1) = E\{x(r_1,t)\}$$
$$\eta_y(r_2) = E\{y(r_2,t)\} \tag{C.2}$$

where $\mathbf{r_1}$ and $\mathbf{r_2}$ are the positions at which the data are taken. From this some correlation terms can be defined:

Auto Correlation $\quad R_{xx}(r_1,t_1,t_2) = E\{x(r_1,t_1)\bullet x(r_1,t_2)\} = \overline{x(r_1,t)\bullet x(r_1,t+\tau)}$

Cross Correlation $\quad R_{xy}(r_1,r_2,t_1,t_2) = E\{x(r_1,t_1)\bullet y(r_2,t_2)\} = \overline{x(r_1,t)\bullet y(r_2,t+\tau)}$ (C.3)

Auto Covariance $\quad C_{xx}(r_1,t_1,t_2) = R_{xx}(r_1,t_1,t_2) - \eta_x(r_1)\bullet\eta_x(r_1) = R_{xx}(r_1,\tau) - \eta_x^2$

Cross Covariance $\quad C_{xy}(r_1,r_2,t_1,t_2) = R_{xy}(r_1,r_2,t_1,t_2) - \eta_x(r_1)\bullet\eta_y(r_2) = R_{xy}(r_1,r_2,\tau) - \eta_x\bullet\eta_y$

Note that all these equations are dimensional, typically mean square values. They can be calculated for two different positions $\mathbf{r_1}$ and $\mathbf{r_2}$, two different times, $\mathbf{t_1}$ and $\mathbf{t_2}$, or a time delay $\boldsymbol{\tau}\mathbf{=t_2\text{-}t_1}$. The *covariance* is the difference of the *correlation* from the mean square value. Instead of the mathematical mean values expressed in Eqs. C-2, they are replaced in the above equations by physically realizable measures. The ensemble measure is replaced by a time history (typically only one) where **T** is considerably less than the infinity suggested in Eq. C-4.

$$\eta_x(r_1) = \frac{\lim}{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(r_{1,}\tau)d\tau = \overline{x(r_1,t)} \tag{C.4}$$

Other metrics used are the *correlation coefficients*. They are second order measures.

Auto Correlation Coefficient $\quad T_{xx}(r_1,\tau) = \dfrac{\overline{x(r_1,t)\bullet x(r_1,t+\tau)}}{\overline{x^2(r_1,t)}} = \dfrac{R_{xx}(r_1,\tau)}{R_{xx}(r_1,0)}$ (C.5)

Cross Correlation Coefficient $\quad T_{xy}(r_1,r_2,\tau) = \dfrac{\overline{x(r_1,t)\bullet y(r_2,t+\tau)}}{\sqrt{\overline{x^2(r_1,t)}\bullet\overline{y^2(r_2,t)}}} = \dfrac{R_{xy}(r_1,r_2,\tau)}{\sqrt{R_{xx}(r_1,0)\bullet R_{xy}(r_2,0)}}$

It can be shown that the range of the coefficients is from –1 to +1, minus related to negative correlations and positive related to positive correlations (Figure C-1). Higher moments can also be defined. Note that these equations are dimensionless, so are independent of the measurement system used and are more useful in learning about sound sources.

In acoustics, the measurement system rejects D.C. values so the means are zero; there is little difference between covariance and correlation. Although *covariance* is the more correct term, use of the more common word *correlation* should not introduce misinterpretations..

Note that all the above equations are based on the *Ergodic hypothesis*; a measure along a time history is equivalent to a measure across a large number of samples at one time.

C.5 Spectral Density

Spectral density is a measure of how much of a given variable resides in a band of frequencies 1 Hz wide. It is a second moment property of the process, so is related to the product of variables. If these products are related to variables such as the square of pressure or velocity, they may be related to power, in which case they may be referred to as *power spectral density*. It is common practice to use the prefix "power" for any spectral density, while here is it properly used only for acoustical power. The other power related variable is the *intensity spectral density* that can be measured directly with proper instruments, or deduced from sound pressure in the far sound field.

As an example, the cross spectral densities are related to the correlations by the transform pair.

$$
\begin{aligned}
S_{xy}(r_1, r_2, \omega) &= \int_{-\infty}^{\infty} R_{xy}(r_1, r_2, \tau) e^{-i\omega\tau} d\tau \\
R_{xy}(r_1, r_2, \tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(r_1, r_2, \omega) e^{i\omega\tau} d\omega
\end{aligned}
\qquad (C.6)
$$

Note that these variables are dimensional. The dummy variable τ can be either positive or negative. Note also that there is no phase information between the various frequencies for either formulation so information is lost, thus it is not possible to reconstruct the record from which the data were taken. In any real system, negative frequencies are indistinguishable from positive frequencies, so the integrals are reduced to one-sided ones, in which the negative frequency domain is folded over.

The mean square value and the correlation can be given in terms of the spectral density.

$$
\begin{aligned}
\overline{x^2(r_1, t)} &= \frac{1}{\pi} \int_{0}^{\infty} S_{xx}(r_1, \omega) d\omega \\
R_{xy}(r_1, r_2, \tau) &= \frac{1}{\pi} \int_{0}^{\infty} S_{xy}(r_1, r_2, \omega) \cos \omega\tau d\omega
\end{aligned}
\qquad (C.7)
$$

The spectral density of a variable, such as source strength, is

$$
S_{QQ}(\omega) = \frac{\lim}{T \to \infty} \left[\frac{1}{2T} Q(\omega) Q^*(\omega) \right]
$$

The above equations can make mathematicians happy. The results are useful to engineers, but are difficult to obtain without a research project. The approach here, when using one-third octave band analysis, is to replace the integrals with summations and only over a finite frequency range. In the **SoundSource** program, each one-third octave band is presumed to be composed of white noise prior to band summations; {C.6.2} discusses converting one-third octave band levels to spectrum levels based on this approximation.

One-third octave band data are easy to obtain and details of these bands are discussed in {C.6.1}.

C.6 Measurement Bandwidths

The need for integration in making approximate analyses of sound sources can be bypassed by using one-third octave band spectra and assume that each band is composed of band limited white noise. Band spectra can be easily measured or modeled based on similar sources. The next section {C.6.1} shows the lower, center, and upper band limits for both one and one-third octave bands in common use. Section {C.6.2} permits a one-third octave band to be reduced to spectrum level based on it being composed of band limited white noise. It then becomes possible to computer model the directivity

$$\begin{aligned} CFreq(BN) &= 10^{BN/10} \\ LFreq(BN) &= CFreq(BN) * 2^{-1/M} \\ UFreq(BN) &= CFreq(BN) * 2^{1/M} \end{aligned} \qquad \text{(C.8)}$$

characteristics of a particular source type. For example, it is possible to obtain the directivity characteristics of a broad-band dipole source above a rigid plane surface at an arbitrary orientation to the surface.

C.6.1 Band Frequency Limits

Typical measurement systems use either one octave or one-third octave bands, while most analysis is on a spectral basis. **B** is the half-width of three common band widths, and ω_c is the center frequency of the band.

The equations to locate the lower, center, and upper frequencies for the standard ANSI bands are shown in Eqs. C.8.

BN stands for the ANSI band number (30=1000 Hz). **M** is the band divisor: 2 for octave bands, 6 for one-third octave bands. Table C-2 shows the results for commonly used frequency bands. The frequencies in parentheses are the commonly used *descriptors* for each band. Note that the center frequencies are based on powers of ten [49, 50] while bandwidths are based on powers of two. In realizable filters, some band overlaps occur, primarily at the higher frequencies, but they are relatively minor. If the level in one band is much higher than that in the contiguous bands, measurement errors can occur.

ANSI Band Number	1/3 Octave Frequencies			1 Octave Frequencies		
	Lower	Center	Upper	Lower	Center	Upper
15	28	32 (31.5)	35	22	32	45
16	35	40 (40)	45			
17	44	50 (50)	56			
18	56	63 (62.5)	71	45	63	89
19	71	79 (80)	89			
20	89	100 (100)	112			
21	112	126 (125)	141	89	126	178
22	141	159 (160)	178			
23	178	200 (200)	224			
24	224	251 (250)	282	178	251	355
25	282	316 (315)	355			
26	355	398 (400)	447			
27	447	501 (500)	563	354	501	709
28	562	631 (630)	708			
29	708	794 (800)	892			
30	891	1000 (1000)	1122	707	1000	1414
31	1122	1260 (1250)	1413			
32	1412	1585 (1600)	1779			
33	1778	1995 (2000)	2240	1410	1995	2822
34	2238	2512 (2500)	2820			
35	2817	3162 (3150)	3550			
36	3547	3981 (4000)	4469	2815	3981	5630
37	4465	5011.9 (5000)	5626			
38	5621	6310 (6300)	7082			
39	7077	7943 (8000)	8916	5617	7943	11234
40	8909	10000 (10000)	11225			

Table C-2. Band limits for two commonly used bandwidths.

C.6.2 Converting One-Third Octave Band Levels to Spectrum Levels

For calculating the interaction of two sources with only one-third octave band spectra, a simple *approximation* is to assume that each band is composed of band-limited white noise. In that way, each frequency within the band can be analyzed separately. Consider the value of **kr** for a one-third octave band centered on 1000 Hz. There is a 25% difference in **kr** within this one band, so significant level differences can occur. For this approximation, the relationship is

$$L_s = L_p(Band) - 10 * \log_{10}(BW)$$

L_s is the white noise spectrum level in the band with the band level, **L_p** and **BW** is the bandwidth (the difference between the upper band limit and the lower band limit) in the particular band. Table C-3 shows the numerical corrections for one-third octave bands.

ANSI Band Number	Nominal One-Third Octave Band Center Frequency	Correction from Band Level to Spectrum Level
15	31.5	-8.6
16	40	-9.6
17	50	-10.6
18	62.5	-11.6
19	80	-12.6
20	100	-13.6
21	125	-14.6
22	160	-15.6
23	200	-16.6
24	250	-17.6
25	315	-18.6
26	400	-19.6
27	500	-20.6
28	630	-21.6
29	800	-22.6
30	1000	-23.6
31	1250	-24.6
32	1600	-25.6
33	2000	-26.6
34	2500	-27.6
35	3150	-28.6
36	4000	-29.6
37	5000	-30.6
38	6300	-31.6
39	8000	-32.6
40	10000	-33.6

Table C-3. Correction from one-third octave band level to spectrum level, based on white noise.

C.7 Band Limited White Noise

Modeling of broad-band sound in this monograph relies on conversion of bands to spectrum levels based on the white noise approximation. Band limited white noise, of band width **b**, can be expressed in the form

$$S_{xx}(r_{1,}, \omega) = S \qquad \omega_c\text{-}b/2 <= \omega <= \omega_c + b/2$$

$$S_{xx}(r_{1,}, \omega) = 0 \qquad \text{elsewhere}$$

It is instructive to look at the structure of this noise. The auto-correlation for it is

$$Rxx(r_{1,}\tau) = \frac{1}{\pi}\int_0^\infty S_{xx}(r_1,\omega)\cos\omega\tau\, d\omega = \frac{S}{\pi}\int_{\omega_c - b/2}^{\omega_c + b/2} \cos\omega\tau\, d\omega = \frac{bS}{\pi}\frac{\sin(\frac{b\tau}{2})}{\frac{b\tau}{2}}\cos\omega_c\tau$$

The auto-correlation coefficient simply divides the correlation by its value at **τ=0**, so the multiplier in the above equation becomes one. Figure C-2 shows this coefficient for a one-third octave band centered at 100 Hz. The sine function acts as a damped envelope around the center frequency. The coefficient oscillates about zero and decays to near zero in about five periods of the center frequency. Note that as the bandwidth *decreases* toward zero, the sine term approaches one, suggesting that the correlation coefficient extends toward infinity as would be expected for a sine wave.

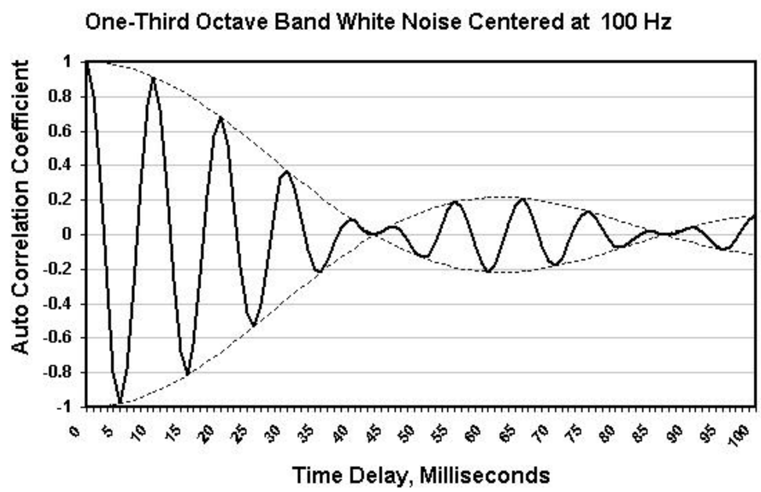

Fig. C-2. The auto-correlation coefficient at 100 Hz for a one-third octave band.

The auto-correlation coefficient is a measure of the relationship of a signal at one time with that at an earlier time. Similar relationships occur *spatially* for turbulent flows. Figure C-3 shows for a white noise spectrum centered at 100 Hz for two bandwidths (one and one-third octave). The broader the bandwidth, the more quickly the coefficient decays. Figure C-4 shows the auto-correlation coefficient for a white noise spectrum centered at 1000 Hz for two bandwidths (one and one-third octave). The coefficient decays more quickly with broader bandwidths. The higher the frequency the more rapidly the coefficient decays. Figure C-5 shows the time for band-limited white noise to lose correlation (a logarithmic decay with frequency).

Key Point: Most of the fluid mechanical models in this monograph for broad-band sound are based on an arbitrary, but variable, number of incoherent sources e.g., {4.9.5}. The results above suggest that smaller turbulent eddies are associated with higher frequencies and smaller correlation distances. To properly model a broad band sound source, say trailing edge noise {4.9.2.2}, each frequency band must have a different number of sources laterally. Since the primary emphasis was on defining the source type and characteristic variables this was considered an unnecessary refinement.

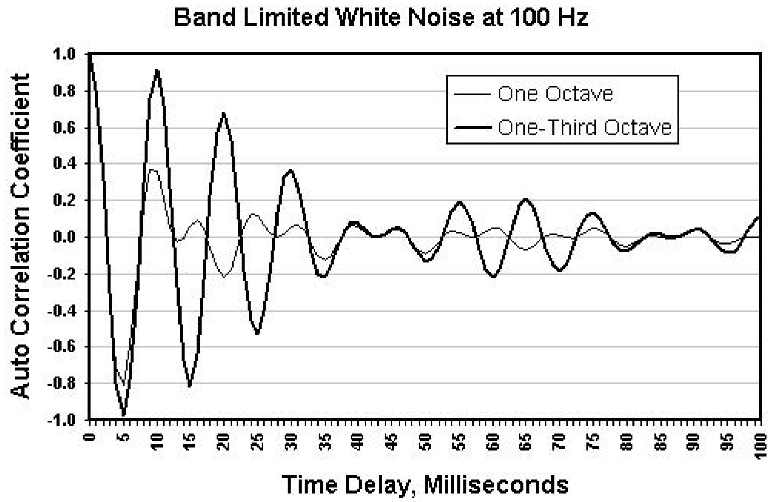

Fig. C-3. The correlation coefficient at 100 Hz for two bandwidths.

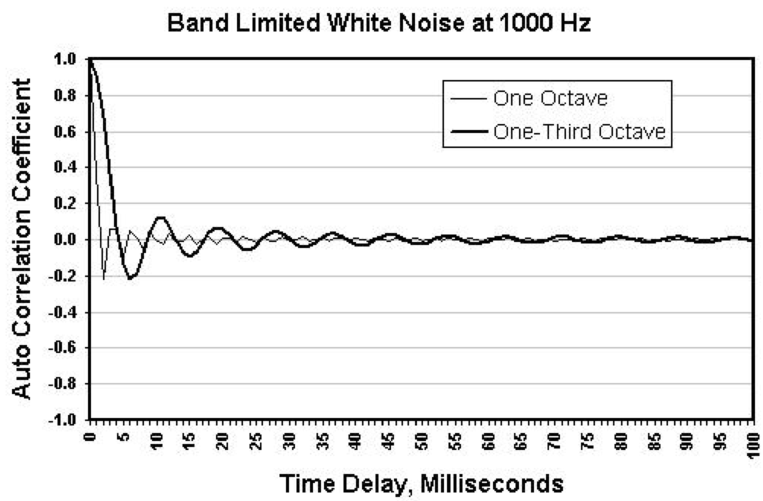

Fig. C-4. The auto-correlation coefficient at 1000 Hz for two bandwidths.

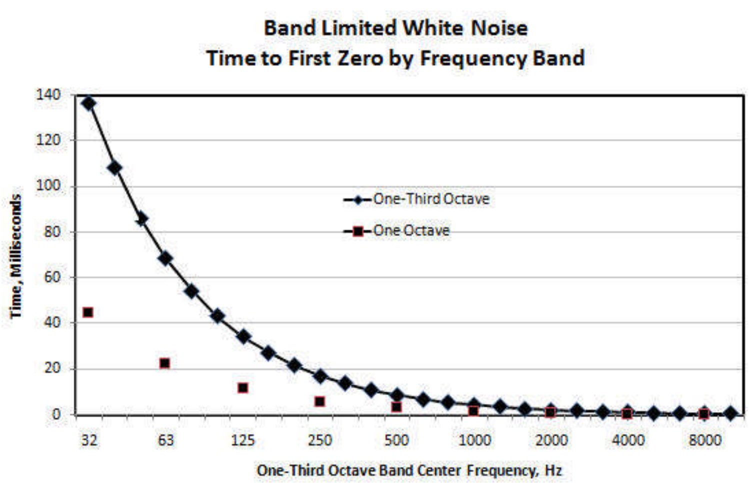

Fig. C-5. The correlation decays rapidly for higher frequencies and broader bandwidths.