Appendix D
Development of the Wave Equation

D.1 The Equations of Fluid Motion

The equations of fluid motion are embodied in the equations of mass, momentum and thermodynamics. They will be expressed in Cartesian tensor notation, as it is more compact. See Appendix B for notation details. The development of these equations is attributed to Claude-Louis Navier (1785-1836) and Sir George Gabriel Stokes (1819-1903) and they are often called the Navier-Stokes equations. Although the developers were concerned primarily with incompressible flows, the equations are sufficiently general to handle compressible flows. These equations express the conditions in an infinitesimal cube of space by calculating the in/out flow and the changes of mass and momentum inside.

D.1.1 The Mass Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho u_i) = 0 \qquad \text{(D.1)}$$

where $\boldsymbol{\rho}$ is the fluid density, **t** the time, $\mathbf{x_i}$ represents the spatial coordinates and $\mathbf{u_i}$ the respective fluid velocities. The subscript **i** varies from 1 to 3 (no relation to $\sqrt{-1}$) . The second term is the net in/out flow of fluid which results in a change of internal density. If zero, the density must be constant, so no sound generation occurs {1.2.1}.

D.1.2 The Momentum Equation

Approximation 1. The fluid is Newtonian. The general form for a normal (Newtonian) fluid is

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial P_{ij}}{\partial x_j} + \varepsilon E_i - \frac{\partial H_{ij}}{\partial x_j}$$

A Newtonian fluid, named for Isaac Newton (1643 –1727), is a fluid where the stress versus rate of strain curve is linear and passes through the origin. The constant of proportionality is called viscosity. Air and water are Newtonian fluids. The fluid response on the left is responding to the surface forces $\mathbf{P_{ij}=p\delta ij -\sigma_{ij}}$. Where **p** is the static pressure, $\boldsymbol{\delta_{ij}}$ is the Kronecker delta function and $\boldsymbol{\sigma_{ij}}$ is the viscous stress tensor. The fluid also responds to an electric field, where $\boldsymbol{\varepsilon}$ is the charge density and **E** the electric field vector. The fluid responds also to the magnetic field where $\mathbf{H_{ij}}$ is the magnetic stress tensor.

Approximation 2. The electromagnetic forces are of no interest, so the momentum equation reduces to

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial u_i}{\partial x_j} \qquad \text{(D.2)}$$

The viscous term on the right includes the dynamic viscosity. Although spherical coordinates are used in this monograph, the complete momentum equation in that coordinate system is excessively long and not illuminating.

D.2 Dynamic Similarity

When studying engineering problems, dynamic similarity is often neglected. Nature knows nothing of the dimensional systems we use, but knows only the ratio of quantities, such as lengths and forces. Important information can be derived by putting the equations of motion in dimensionless form and from that dynamic similarity ratios can be derived. This is done by relating the physical variables to reference quantities such as shown in the list below. The values subscripted with a 1 are now dimensionless. By convention, the dimensionless density is given the symbol **s**. Two of those reference quantities (**L** and **U**) can be called *characteristic* numbers and constitute what may be called *scaling rules*. There is no loss in generality if Eqs. D.1 and D.2 are reduced to one-dimensional Cartesian coordinates where only motion along the **x** axis occurs. These equations are shown in Eqs. D.3.

$$\rho = \rho_0 (1+s)$$
$$p = p_0 (1+p_1)$$
$$u = U u_1$$
$$x = L x_1$$
$$t_1 = \omega t$$
$$c_0^2 = \frac{\gamma p_0}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial x^2} \qquad \text{(D.3)}$$

Substituting the new variables into Eqs. D.3, the equations are now expressed in dimensionless form.

$$2\pi St \frac{\partial (1+s)}{\partial t_1} + \frac{\partial}{\partial x_1}\left[(1+s)u_1\right] = 0$$
$$2\pi St \frac{\partial u_1}{\partial t_1} + u_1 \frac{\partial u_1}{\partial x_1} = -\frac{1}{\gamma Ma^2 (1+s)}\frac{\partial p_1}{\partial x_1} + \frac{1}{\mathrm{Re}(1+s)}\frac{\partial^2 u_1}{\partial x_1^2} \qquad \text{(D.4)}$$

The kinematic viscosity **ν** has been introduced. Three important dimensionless parameters are now obvious; see {A.2.2} for more details on them.

$\mathrm{Re} = \frac{UL}{\upsilon}$ **Reynolds number.** The ratio of the steady inertial forces to the steady viscous forces.

$St = \frac{fL}{U}$ **Strouhal number**. The ratio of unsteady inertial forces to steady inertial forces.

$Ma = \frac{U}{c_0}$ **Mach number**. The ratio of the steady speed to the speed of sound.

These numbers are extremely useful in understanding the nature of sound sources and may be considered essential for evaluating dynamic similarity. If the characteristic length **L** and the characteristic speed **U** are chosen properly, much can be learned about the physics of a particular problem. Inversely, information obtained from a sound source can be used to get good estimates of what these two characteristic variables might be. As is shown in the chapters, these variables are best thought of as *local* ones, e.g., the steady speed of a jet varies with distance from the nozzle.

D.3 The Equation of State

In general, the equation of state for a gas can be expressed in the form

$$p = p(\rho, e)$$
$$dp = \left.\frac{\partial p}{\partial \rho}\right|_e d\rho + \left.\frac{\partial p}{\partial e}\right|_\rho de$$

Where **p** is the pressure, **ρ** is the density, and **e** is the entropy.

***Approximation 3*.** The motion is isentropic (adiabatic and reversible) so we can define a sound speed.

***Approximation 4*.** It is a perfect gas: $p=\rho RT$ and $p=K\rho^\gamma$

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_e$$
$$c^2 = \frac{\gamma p}{\rho} = \gamma RT$$
$$c^2 = \frac{\gamma p_0(1+p_1)}{\rho_0(1+s)} \qquad \text{(D.5)}$$
$$p_1 = \frac{p-p_0}{p}$$
$$s = \frac{\rho-\rho_0}{\rho_0}$$

T is the *absolute* temperature, **K** is a constant and **γ** is the ratio of specific heats.

A perfect gas is one in which the molecules have elastic collisions. It is a good approximation to the behavior of air under many conditions. Émile Clapeyron (1799-1864), August Kronig (1822-1879), and Rudolf Clausius (1822-1888) helped to develop the relevant equations.

***Approximation 5*.** The magnitude of the pressure fluctuations $\mathbf{p_1}$ and those of the density **s** are so small relative to one that they can be neglected. The sound speed is now

$$c_0^2 = \frac{\gamma p_0}{\rho_0} = \gamma RT_0 \qquad \text{(D.6)}$$

A typical value of **γ** is 1.4 and that for **R** is 53.3 ft-lb_f/lb_m°R. Note that all of the above approximations require that the sound field be a small perturbation on the static condition. This is not always true {2.2.6}.

Note also that in order to have a propagation speed, and therefore sound, the medium must be elastic (*the density must vary*). Since both solids and fluids are elastic, most (but not all) time varying motions will generate sound. There are many cases where the sound created is un heard because it is either too low in level or outside the frequency range of listeners {1.6}.

D.4 Simplifying to the Wave Equation

The sound field is a subset of the fluid motion, so the acoustic wave equation must be imbedded somewhere in the equations of fluid motion. How deeply is it buried? In the general case, cross differentiating the continuity and momentum equations (Eqs. D.1 and D.2) yields

$$\begin{aligned}
&\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial t}(\rho u_i)\right) = 0 \\
&\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial t}(\rho u_i)\right) + \frac{\partial^2}{\partial x_i \partial x_j}(\rho u_i u_j) = \frac{\partial^2 P_{ij}}{\partial x_i \partial x_j}
\end{aligned} \qquad \text{(D.7)}$$

The development of the acoustic wave equation from these equations is given in a number of texts. A particularly illuminating approach was done by Sir Michael Lighthill (1924-1998) who developed the theory of sound generated aerodynamically [12,13]. For completeness, his form of the wave equation is presented below, but it is still too general.

$$\begin{aligned}
&\frac{\partial^2 \rho}{\partial x_i^2} - \frac{1}{c_0^2}\frac{\partial^2 \rho}{\partial t^2} = -\frac{1}{c_0^2}\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \\
&T_{ij} = \rho u_i u_j + P_{ij} - c_o^2 \rho \delta_{ij}
\end{aligned} \qquad \text{(D.8)}$$

Rather than repeat the general development and with no loss of generality, the wave equation for the one dimensional case is developed. Eqs. D.4 suggest approximations that can be made.

Approximations 3, 4, 5**.** See previous pages.
Approximation 6. The Reynolds number is large enough so that viscous terms can be neglected. See {2.3.7}.
Approximation 7. The Mach number is small enough so that the pressure gradient term can be retained.
Approximation 8. The third term in the continuity equation is much smaller than the second.
Approximation 9. The second term in the momentum equation is much smaller than the first.

With these approximations, Eqs. D.3 are reduced to the following. The terms in red are deleted.

$$\begin{aligned}
&\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial su}{\partial x} = 0 \\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial x^2} = \frac{1}{\rho_0}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial \rho} = c_0^2\frac{\partial s}{\partial x}
\end{aligned} \qquad \text{(D.9)}$$

Cross differentiating the two equations yields the wave equation for density fluctuations.

$$\frac{\partial^2 s}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 s}{\partial t^2} = 0 \qquad \text{(D.10)}$$

The condensation is related to the other physical variables {A.3} so the wave equation can be made to apply to each of them.

D.5 Validity of Approximations

It is necessary to show *a posteriori* under what conditions the approximations are valid.

Approximation 1. The discussion in this monograph applies only to air so that this approximation is applicable. Slight deviations from a Newtonian fluid may not create problems, since the influence of viscosity is sufficiently small to be neglected.

Approximation 2. The discussion in this monograph does not apply to extensive electromagnetic fields. If the effect is actual but contained *within* the hypothetical surface, it has no influence on the wave equation.

Approximation 3. If the motion is small enough and quick enough, the motion will be adiabatic and reversible. An abstract from Eqs D.5 is given on the right. From Table 4-1, at a level of near 88 dB, p_1=4.72x10^{-6} and s=3.43x10^{-6}, more than sufficient to meet the approximation.

$$c^2 = \frac{\gamma p_0 (1+p_1)}{\rho_0 (1+s)}$$

Approximation 4. The discussion in this monograph is restricted to air in the volume outside the hypothetical surface, so the perfect gas approximation is good.

Approximation 5. The magnitude of the pressure and density are sufficiently small to neglect nonlinear terms. See Approximation 3.

Approximation 6. The Reynolds number of the steady flow in most situations is sufficiently large so that viscous effects can be neglected. See {2.3.7}.

Approximation 7. The Mach number, if subsonic, defines the boundary between essentially incompressible flow and compressible flow (regardless of the sound field). *If the hypothetical boundary* is at an appropriate distance (outside any flow field), the *local* Mach number is sufficiently small to make the approximation valid.

Approximation 8. The inequality $|su| \ll |u|$ is easily met if Approximation 3 is met.

Approximation 9. The inequality $\left|u\frac{\partial u}{\partial x}\right| \ll \left|\frac{\partial u}{\partial t}\right|$ is met if $M \ll 1$. Although the characteristic speed was not restricted in forming Eqs. D.4, the speed for this case is that of the so-called *acoustical particle* motion, which implies a restriction that the medium be at rest.

The wave equation is normally satisfied for normal sound levels experienced outside the hypothetical surface. Next, it is worthwhile to consider where that surface should be by examining the various fields that surround a source.

D.6 Regions Around a Sound Source

It is instructive to look at the influence of the various approximations as the distance from a small sound source is decreased: the monopole model is used as an example. Far from the source, the wave equation is valid, but as the source is approached, the approximations needed to make that equation valid are violated, so the equations used for analysis must change. Those changes are listed below and are shown graphically in Figure D.1. It is important to note that the equations of motion (Eqs. D.1) *always* apply everywhere outside a surface within which chemical or nuclear reactions occur. The relationships shown below are just those used to analyze the *dominant* features of the fluid motion.

It is presumed that the medium, through which the sound passes, is at rest.

D.6.1 The Sound Field Region

The momentum convection term can be neglected as noted in Eqs. D.10. The local Strouhal number is useful and is based on the distance and local particle speed. The local Mach number is also defined the same way and the subscript l is applied to denote the difference.

Using Eqs. 3.7, the dependence on distance of the two local parameters and the condensation are shown on the right. The influence of the density change term in the continuity equation becomes more prominent as the field becomes more "acoustical" with greater distance and decreases with reduction of distance. The requirement for the condensation to be small is shown in the relationship on the right; there is an upper limit on both frequency and source strength for the wave equation to be valid. When it is, the relevant equations are

$$St_l \propto r^2, Ma_l \propto r^{-1}, s \propto r^{-1}$$

$$\frac{fQ_m}{2rc_0^2} << 1$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\nabla p$$

$$p = \rho_0 \frac{\partial \phi}{\partial t}$$

$$\Box^2 \phi = 0$$

The pressure gradient has been generalized to avoid geometric complexity.

D.6.1.1 Geometric Far Field. r>>h.

Considering the interaction of two point sound sources in Chapter 3, the separation distance between them was given as **2h**. An approximation where $kr_1 \cong k\left(r - h\cos\theta\right)$ was made to simplify the calculations (Eqs. 3.18). Two aspects of that approximation are important: relative magnitude and relative phase. It is common to assume that the distance factor for each source is the same. If the direction is along the line between the sources (worst case) and $r = 10h$, the magnitudes of the two inverse distances are 20% different. The phase differences created by the above approximation are only important at high frequencies. It is important for the measurement point to be as large a multiple of the distance between the sources as possible.

D.6.1.2 Sound Far Field. kr>>1.

The ratio of the distance to the sound wavelength is much greater than one. Examining the data in Chapter 3, the reactive component of the intensity vector is 10% of the radial component at kr=10. The pressure and velocity are essentially in phase. This has importance for deducing intensity from sound pressure measurements. The actual distance to meet this recommendation is a function of frequency. The distance may be chosen based on the equation on the right; the second term is for normal temperatures. Table D.1 below provides some numerical results.

$$r = \frac{1.59c_0}{f_{\min}} \approx \frac{1793}{f_{\min}}$$

Frequency, Hz	**50**	**100**	**500**	**1000**
kr=10	36 Feet	18 Feet	43 Inches	22 Inches

Table D.1. The distance to the far sound field depends on frequency.

D.6.1.3 Sound Intermediate Field. kr≈1.

The ratio of the distance to the sound wavelength is near one. Both reactive and resistive terms must be included in the equations. The pressure and velocity are approximately at 45 degrees, suggesting the balance between the compressible and incompressible aspects of the motion.

D.6.1.4 Sound Near Field. kr<<1.

The ratio of the distance to the sound wavelength is much less than one. The reactive components are significant and the resistive component (the sound field) is less significant. The pressure and velocity are nearly at 90 degrees to each other, suggesting the dominance of the incompressible aspect of the flow. The sound field is buried inside the incompressible flow field. If kr=0.1, the sound is only 10% of the incompressible motion.

D.6.2 The Hydrodynamic (Quasi-incompressible) Flow Region

The incompressible aspects of the flow are dominant in this region. The momentum of the flow is more significant. The validity of the wave equation is now in doubt. The buried sound field is sufficiently small that it is often neglected in favor of the presumption that the density is constant as shown in the equations on the right.

$$\rho = \rho_0$$
$$\nabla^2 \phi = 0$$

D.6.2.1 Hydrodynamic Far Field. kr<<1.

This field overlaps the near sound field, so either approach, the compressible or incompressible, can be used. Most of the pressure is used to accelerate the fluid. If the sound speed is set to infinity, both the pressure and velocity, which are independent of the sound speed, are unaffected, the condensation becomes zero, suggestive of incompressible flow.

D.6.2.2 Hydrodynamic Intermediate Field.

The flow is sufficiently incompressible that the density change has little influence on the flow field. The momentum of the flow is sufficient that the wave equation is an invalid description of the flow. Although it can be neglected in any calculations, the sound field still exists.

$$(u \bullet \nabla)u + \frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\nabla p$$
$$p + \frac{1}{2}\rho_0 u^2 = \rho_0 \frac{\partial \phi}{\partial t} + Const$$

D.6.2.3 Hydrodynamic Near Field.

The incompressible components totally dominate the motion so it is reasonable to treat the flow as completely incompressible. Two relevant equations are shown on the right.

$$(u \bullet \nabla)u = -\frac{1}{\rho_0}\nabla p$$

$$p + \frac{1}{2}\rho_0 u^2 = Const$$

D.6.3 The Compressible Flow Region

Closer to the source the density fluctuation becomes so large it can no longer be treated as a constant. This region is generally treated as *high speed compressible flow*. Two relevant equations are shown on the right.

$$(u \bullet \nabla)u = -\frac{1}{\rho}\nabla p$$

$$\frac{\gamma - 1}{\gamma}\frac{p}{\rho} + \frac{u^2}{2} = Const$$

D.6.3.1 Subsonic Field. M<1

At high subsonic Mach numbers, the equations of motion still allow for sound to propagate to other parts of the medium. For this case the acoustic particle motion is no longer small and is better interpreted as an oscillatory flow speed in calculating the Mach number. The density change is sufficiently large that the speed of sound equation is no longer valid within this region. This occurs with large amplitude motions where the wave shape is progressively modified toward a shock wave {2.2.6}. Low power explosions are examples of subsonic transients of high amplitude.

D.6.3.2 Supersonic Field. M>1.

At supersonic Mach numbers, the equations of motion do not allow sound to propagate to other parts of the medium. Shock waves are common expressions of this motion. High power explosions or nuclear shock waves are examples.

D.6.4 The Reaction Region

Within this region, either chemical or nuclear reactions occur. The usual Navier-Stokes equations are not applicable. Depending on the nature of the reactions this region can replace, or overlap, the compressible and incompressible regions. At one extreme is a propane torch where its source magnitude is small enough so that the hypothetical surface can be reasonably close. At the other extreme is the H-bomb, the hypothetical surface is likely to be incinerated at any reasonable distance.

D.7 Key Points

Looking at the details of a sound source shows the potential complexity of mathematically approaching the source. When the source is treated as a black box with a hypothetical surrounding surface outside of which the wave equation for a still medium is valid, much useful information can be gained. The challenge is to choose the correct distance of that surface from the source. The discussion above shows that this can be a difficult choice and the intent here is make the reader aware of what items must be considered to make that choice. If any measurements are made within that bounding surface, substantial errors of level and spectrum can occur. No attempt has been made to look at a medium in motion. That adds (or subtracts) a steady speed to the sound speed, with resulting time delays, level, and Doppler shifts. Most ordinary sound problems occur at low enough Mach numbers where these effects are not significant.

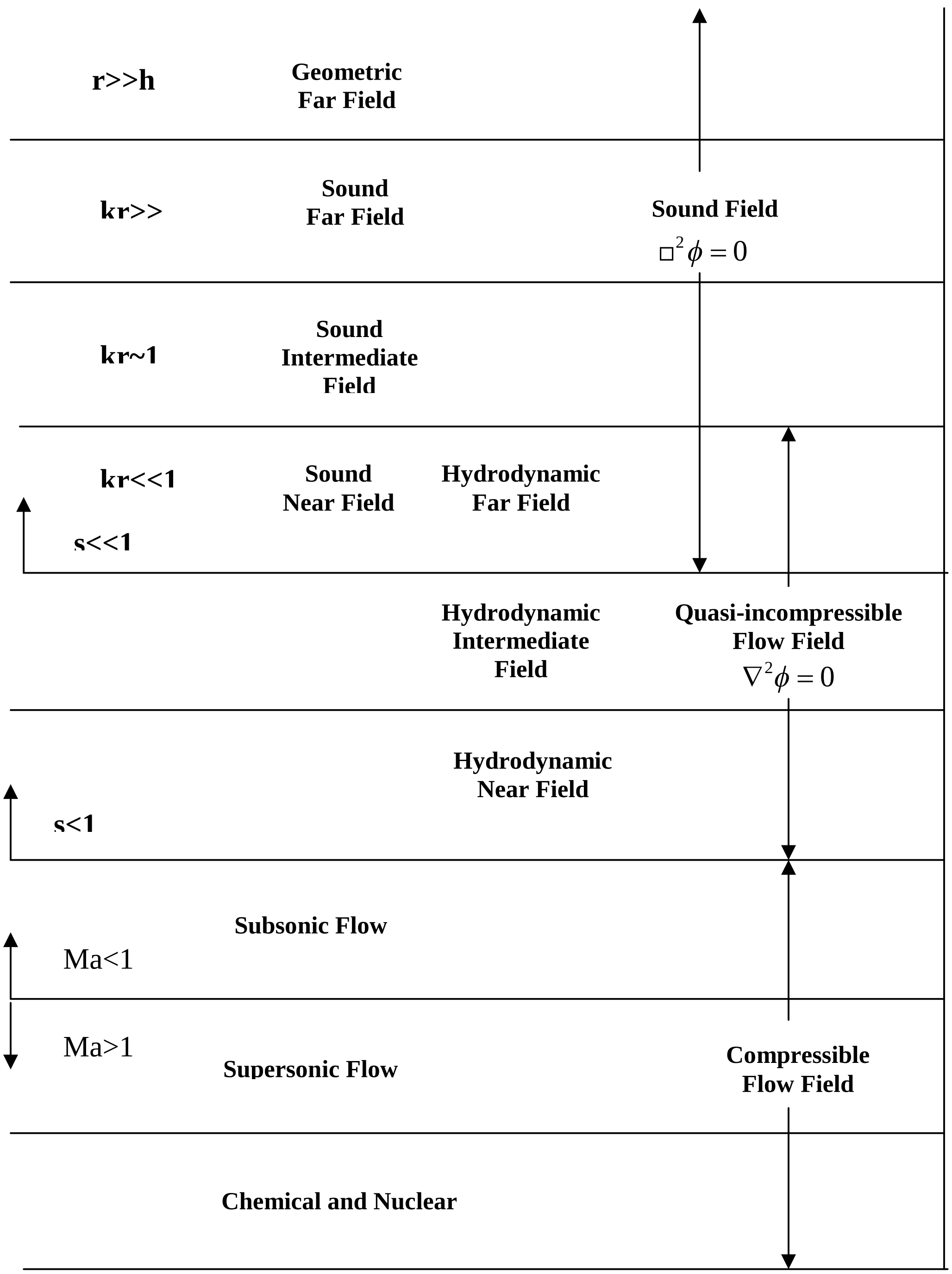

Fig. D-1. Regions of fluid motion around a sound source.