Tools for Analyzing Sound Sources

Dynamic similarity concepts and the three basic source types can be used to put a knowledge framework around noise sources.

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Foreword

This monograph is directed toward engineers, particularly mechanical engineers, who are asked to evaluate the sound output of a product without prior training in acoustics. The standard approach to understand their problem for them is to collect papers and books that may be relevant to the problem with the hope that some enlightenment might result. However, most peer-judged published papers focus on methods to solve a specific problem to the exclusion of a more general approach. There are ample number of excellent texts [1-11] that tend to cover the entire field of acoustics, so are less specific than this monograph. They are valuable, of course, for someone wanting more depth in the subject and has time to pursue study.

The standard approach to solving their problem is to throw a box around it, or move it far away! Unfortunately, there are many sound sources that cannot be boxed or moved away; further, boxes often create heat problems. The emphasis in this monograph is to look at the sound source itself to uncover methods by which the sound output can be diminished. The successful quieting of airliner jet engines is an example of this approach. To do this, a person faced with a sound problem needs a straightforward and general way to place a knowledge framework around a sound source. The expression “knowledge framework” refers to a means of identifying the important characteristics of a source that lead to estimates of sound pressure and, sound power, and how those characteristics vary with size and speed changes. It is the intent of this monograph to do just that. This is accomplished by reducing the three subject areas of acoustics (source, path, and receiver) just to the source, and more specifically to the sound generation process in quiescent air. It excludes sources in motion as most community noise sources are stationary. First, the theory for three fundamental types of sound sources [12, 13] is given extensive treatment in order to expose the parameters that help build the knowledge framework. Then a number of actual sources are studied for each of the theoretical models; it is surprisingly common how many sources to fit into one or the other of the three theoretical models. Geometric and dynamic similarity principles are used to identify lengths and speeds characteristic of the source. The word “characteristic” is used here to imply those lengths and speeds that define the motion of the sound source. The characteristic variables are then used to estimate how sound pressure and sound power depend on them. In each of the examples, focus is on identifying the characteristic variables in order to suggest approaches to identifying them in other sources that might be encountered by an engineer. Emphasis is on single frequency (pure tone) sources since they are always the most objectionable. A number of whistle sources are discussed; they are generally associated with feedback mechanisms that are common but unfamiliar to product engineers.

The theory of the model sources is based on the acoustical wave equation in air. One of the nice things about the wave equation is that it is linear. One of the bad things is that it is derived from the non-linear equations of fluid mechanics. The limits of the linear theory are not always clearly exposed. An appendix is provided showing the number of approximations that need to be made to convert the non-linear equations to the wave equation. Often for real sources, the sound (linear) field is buried within the (non-linear) flow field, making both estimates and measurements of sound difficult. The theoretical models, and deductions about sound pressure and sound power, are applicable only at a certain radius from the source. Other appendices are added to provide a convenient location for data, useful formulas, dimensional variables, and dimensionless numbers often used in acoustics.
On occasion sound sources are ignored because the frequency of the maximum level is above or below the frequency limits of human hearing, or the level is below the human threshold. Unfortunately, when product design changes are made, the level or frequency may move into a part of the hearing envelope that causes objection. Often, the impact of the change in noisiness heard by listeners is not taken into account. This aspect is addressed in the monograph.

Sound sources are created either by the motion of a solid or by the motion of a fluid. One of the problems in analyzing the latter case is sound created by turbulent flows. The relationship of the sound generated by one volume of flow to that generated by another must be taken into account. Turbulent flow volumes have a distance, the correlation length, beyond which two volumes act as independent sources. Correlated sources add pressures linearly, while uncorrelated sources add in a mean square sense (energy). Sound level estimates require knowledge of the correlation which is determined by the size and distance between adjacent eddies. If the correlation distance is known, it is possible to model an overall source as a sum of independent sources, as opposed to the much more difficult, but exact, method of integration. Reducing a problem to a sum of independent sources allows for computer modeling. Unfortunately, correlation lengths are known in only a few situations, so an alternative approach is to model a broadband random sound source as the sum of independent sources based on assumed correlation lengths that are then adjusted to best match measurements.

Robert Chanaud
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Chapter 1
Preliminaries

1.1 The Approach

The field of acoustics has expanded enormously over the last hundred years. It now includes such diverse subjects as sonar, sonography, noise control, audio, and its original meaning: speech and hearing. Each of these areas can be subdivided into source, transmission, and receiver. This monograph is restricted to sources in air and is intended for persons with a technical background in fields other than acoustics, but with the task of understanding the noise aspects of a particular product. The approach is to develop physical intuition about sound generation, particularly from a fluid mechanical viewpoint, so that the important factors of a particular problem can be determined more readily. This is accomplished by first developing theoretical models of the three fundamental types of sound sources. Mathematics is used only to elucidate the physical principles underlying the sound generation by these three sources types. The models are applied to numerous examples of actual sound sources to show how useful information can be extracted about them. The intent is not to provide specific solutions to any particular problem, but rather to show how to approach an unknown sound problem.

The concepts of similarity are used to extract information about the how the sources depend on changing conditions. Dynamic similarity is based on the fact that nature knows nothing of the dimension systems we use; it cares only about ratios of things. Geometric similarity might be considered part of dynamic similarity. These principles are used to scale the three fundamental sources.

Since sound can be created by nuclear or chemical reactions, solid or fluid motion, the approach is to surround the source with a hypothetical surface and be concerned only with the motion outside that surface. The source becomes a “black box” about which information is needed. The surface is positioned such that beyond it the classic wave equation is valid and the sound sources on that surface can be reduced to a distribution of the three fundamental types.

Sound sources can divided into two categories: those in which the sound is a byproduct of source motion and those in which the sound is an integral determinant of source motion. The latter category is significant in that it often results in single frequency sound (whistles, tones) that is more offensive to listeners than broad-band sound.

1.2 Making Sound

One of the fortunate results in the development of science and engineering is the unification of many seemingly diverse fields. When we concern ourselves with the mechanical transport of things from one place to another, we find that we need only be concerned with three basic quantities: mass, momentum, and energy. Nature seems to have provided only two transport means: through direct contact of objects or through a transfer process between objects (action at a distance). For example, hit a billiard ball (or molecule) into another ball and the momentum and energy are transferred by direct contact. A moving fluid carries its momentum by convection on a macroscopic scale, and diffusion on a microscopic scale. The transfer process is called radiation on a macroscopic scale and conduction on a microscopic scale. In
each of these cases, it is necessary to consider what is transferred. The primary concern here will be mechanical radiation, particularly sound, that transports momentum and energy. Mass can be transferred with intense sources (acoustic streaming), but is not addressed here.

1.2.1 The Mathematical View

The motion of normal fluids (air) is governed by the Navier-Stokes equations. Figure 1-1 shows the nice analogy between these equations and those of electrodynamics, solid mechanics, and quantum mechanics. In each case wave motion is involved.

Mechanical waves can be sub-divided into two types based on the mass continuity

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \\
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \\
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0
\]

(1.1)

equation \{D.1\} and can be expressed in three forms:
Consider a small cube. The right hand terms in the above equation account for the inflow and outflow of mass from that cube. If the net flow is not zero, the mass in the cube must either increase or decrease, resulting in changes in the density within the volume.

All fluids are inherently compressible, even cosmologists talk of primordial sound during the big bang. Since the density can always change in a compressible fluid, so the right hand terms in the equation must not be zero when this happens. Mathematically it can be expressed as \( \nabla \times \mathbf{u} = 0 \); the curl of the velocity field is zero. When this condition is met, it is possible to define a velocity potential \( \phi \) where \( \mathbf{u} = -\nabla \phi \). As will be seen later, a solution for the velocity potential permits all relevant physical variables to be defined from it.

The subject of hydrodynamics makes the assumption that the compressible aspects of the fluid are negligible. Mathematically, incompressible motion is characterized by the expression \( \nabla \cdot \mathbf{u} = 0 \); the divergence of the velocity field is zero. When this condition is met, the density is constant and it is possible to define a streamline; a line that is tangent to the fluid velocity at each point. In many cases, insight about the sound field directivity can be obtained by visualizing the streamlines near the source.

The important point here is that both types of motion are generally present in any situation, but only one contributes to the radiated sound. Since the time rate of change can be anything, a frequency spectrum is implied. Linearization of the Navier-Stokes equations yields the wave equation and a speed of transport with respect to the fluid: the speed of sound. The development and limitation of the wave equation is discussed in Appendix D. This is important since the location of the hypothetical surface is determined by the limitations.

### 1.2.2 The Electrical View

Any mechanical motion of an object (the "source") couples into the surrounding medium by virtue of the fact that the contiguous molecules must have similar motion. Using electrical terminology, the coupling must be partially resistive so that the power of the source is "dissipated" and transferred elsewhere. This part is associated with the compressible motion around the source as noted in the previous section. The coupling may be partially reactive, accounting for energy storage and is associated with the incompressible motion around the source. The two components of the coupling sum to the total impedance that the source "sees".

### 1.2.3 The Fluid Mechanical View

The time-varying pressure and viscous forces exerted by a finite sized body on the surrounding fluid result in a flow around the body. The elasticity of the surrounding fluid allows some of the pressure forces exerted by a finite sized body to compress it as well as accelerate it. If the compression is small and time varying, it can be called "sound". If the compression is large, high amplitude sound waves (shock waves) are formed \( \{2.2.6\} \). In this case, the compression may be so large that high speed compressible flow concepts are used, such as those used in the design of high speed aircraft. One limitation to the wave equation is that the waves must be small amplitude. The important point here is that if the source creates shock waves their form is retained outside the surrounding surface so the boundary must be extended until the waves are considered small amplitude.

If the forces that accelerate the fluid are large, the motion is considered hydrodynamic and the compressible components are neglected despite the finite value of the sound speed. That
is, the sound field is buried within the other fluid motion. As an example, placing your ear near the open window of a moving automobile results in a lot of “sound”, but is it? The incompressible motion dominates any compressible motion and a person further away hears nothing. In actuality, the air passing over your ear will create density changes due to the obstruction, so sound will be created, but it will be so minor it is not detected. It is important to note that there are many cases when the dominance of incompressible motion leads to the assumption that no sound is created. What is really implied is that the sound is outside the envelope of human hearing or unable to cause resonant responses in nearby structures. It is valuable to understand the process of making sound so that no surprises occur when size or speed changes are made in a product. An example relevant to this issue is a vertical branch sticking out of a moving stream; typically nothing is heard. We know it will move in oscillatory fashion due to fluid forces, but consider the case where it stands still. Does the flow over it make sound in both water and air? The answer is yes; air and water speed results in an Aeolian tone {4.10.1.1}, however, the frequency and magnitude are outside the human hearing envelope.

### 1.2.4 The Simple View

A simple way to view sound production is to oscillate a flat plate. If the motion is perpendicular to the surface, the air immediately above the plate must signal the further air to move away and this results in density changes that signal the other molecules to move; sound is created. If the motion is in the plane of the surface, does it create sound? No. Energy is transferred to the fluid by viscous forces (shear stresses) only and no density changes occur. Note that the plate here is flat, not wavy. A simple tool to examine a complex noise source is to put a hypothetical surface around it and look at the direction of motion of the source surface. It will provide a clue about which areas make sound.

To transfer power by time-varying mechanical motion there must be some force, which can be expressed as a force per unit area (pressure or viscous shear) times an area, and there must be a response. In acoustics, we use velocity to describe the response to that force. The power $W$ transferred can be expressed symbolically as $W = \langle F \cdot U \rangle$ where $\langle \rangle$ is a time averaging symbol and the $\cdot$ represents the dot product of two vectors, essentially the cosine of the angle between the two variables. When the two quantities are co-linear (for single frequencies it implies they are in-phase), the power transfer is maximized and entirely resistive. When the dot product is zero, there is no power transfer and the two factors are at quadrature (for single frequencies it implies they are ninety degrees out of phase).

Based on this simple concept, there are several means for reducing sound generation:

1. Reduce the force to the source.
2. Reduce the velocity (or RPM).
3. Reduce the area in contact with the surrounding medium (make the source smaller)
4. Reduce the dot product (change the phase relationship or correlation)
5. Reduce the response (reduce the impedance of the surrounding medium)

These suggested methods are generally useless unless there is more understanding about the sound generation process. That is one goal of this monograph.
1.3 Types of Sound Sources

There are two general types: those whose dimensions are small with respect to a wavelength, and those that are not. It has been shown [33, 34] that sources with extensive surfaces can be composed of a distribution of many sources that are small. Although emphasis is on small sources, certain aspects of sound generation can be elucidated by discussing plane sources (Chapter 2) because the geometry and mathematics are simpler.

Geometry plays a role in the sound radiated. One or two dimensional radiation is related to plane sources, and spherical coordinates are used for point (or small) sources. Cylindrical coordinates are not discussed in this monograph although there are several important linear sources such as road travel noise or transmission line corona noise. A small vibrating sphere is an example of a small source. Figures 1-2 to 1-5, show the several ways a sphere move. The figures show only one phase of an oscillating motion.

In Figure 1-2, the sphere expands and contracts uniformly in all directions as would a balloon. There is only one coordinate: radial. The outer dashed circle represents a fixed spherical surface surrounding the sphere. The fluid on that outer surface must move outward and inward causing a net mass flow across that surface. The center of mass of the expanding sphere remains fixed, but the volume changes. This source is associated with changes in volumetric or mass flow rate and is called a monopole. It is the subject of Chapter 3.

In Figure 1-3, the sphere is rigid and moves bodily to a new position. The motion is in a specific direction, so three coordinates are needed to specify it. The center of mass obviously moves due to some applied force. The volume of the sphere is unchanged. The motion of the surrounding fluid on the outer dashed circle is now different. The fluid on the right must move out of the way and the fluid in the wake must follow the sphere. In the two other quadrants, the fluid moves toward the wake. This source is composed of two monopoles of opposite sign very close to each other. Since they are of opposite phase there is some cancellation suggesting that this source is less efficient at creating sound than the monopole. It is associated with force and momentum and is called a dipole. It is the subject of Chapter 4.

In Figure 1-4, the sphere is stretched in one direction by two in-line opposing forces. The volume of the sphere remains unchanged. The center of mass remains unchanged,
so there is no net force acting on the sphere. This motion is the result of longitudinal stresses being applied to it. The motion of the surrounding fluid on the outer spherical surface is different from the earlier examples. The fluid to the right and left must move outward while the fluid in the other two quadrants must move inward. This source can be considered to be composed of two dipoles in-line. Since they are in oppose phase, it is expected that this source is even less efficient than a dipole in creating sound.

In Figure 1-5 the sphere is distorted by laterally opposing forces (lateral shear). The two dipoles that compose the source are in tandem as opposed to in-line. The volume and center of mass remains unchanged. The motion of the surrounding fluid on the outer spherical surface is different once again. The fluid must move away from the two protruding sides and toward the other sides.

These two sources are associated with stresses in the fluid and are called quadrupoles. They are the subject of Chapter 5.

It might be interesting for a mathematician to change the strength and orientation of one force relative to the other, and thereby create a unified monopole-dipole-quadrupole source. Such sophistication is not necessary here.

The seminal work by Lighthill [12, 13] has shown that sound generation in fluids is the result of integration over an area of a distribution of the three types of sources. This is a formidable task in most cases, so one aim of this monograph is to create approximations by modeling the integration as a summation. It is surprising how much beneficial information can be obtained with this simplification.

It should be noted that the three sources are really the first three modes of vibration of a sphere. Higher order modes occur in solid objects, such as compressor housings, but when the sound radiation is considered, these higher modes can be constructed from the three lower modes [12, 13].

### 1.4 Categories of Sound Sources

#### 1.4.1 Category I

The sound from a Category I source is primarily a by-product of source motion. In every case of coupled bodies (a source and the surrounding medium), the response of one to the excitation of the other has a back reaction on the exciter. The resistive and reactive loading (impedance) of air on a solid body is very small (density of air is about 0.075 lb/ft\(^3\), that of steel is 490 lb/ft\(^3\)). That is why most solid vibrations are analyzed as if the material were in a vacuum \{2.3.5\}. Under water the story is different; fluid loading is very important. Loading is also important when the sound is generated by the fluid motion itself (jet noise). In many cases, the loading on the source motion can be taken into account and the output calculated, but in each case it does not control source motion. The characteristic speed is the one that determines the sound power and contributes to frequency determination. The characteristic length may contribute to the sound power as well as contribute to frequency determination.
1.4.2 Category II

The output (sound or fluid motion) is an integral determinant of source motion. In many important cases, linear thinking (small cause = small effect) is fallacious. The output can feedback to the source and control it. The oscillations are fed back to the input of the amplifier part of the process in proper phase and amplitude to take control. The analogy with a feedback controlled electrical oscillator is exact. A common example of an electrical/acoustical feedback loop is the squeal heard in an auditorium when the sound from a loudspeaker gets back into the microphone to be further amplified. The sound from the loudspeaker has a strong influence on its own sound generating motion. The basic requirements for a feedback controlled sound source are: (1) a source of power; (2) an amplifier which can convert the steady power to time varying power; (3) an initial disturbance which supplies the oscillations to be amplified; (4) a means of generating sound or other oscillatory fluid motion; and (5) a means for feedback of that oscillatory motion to the input of the amplifier. Such sources are often referred to as "whistles" or "tones" because they generally have a periodic waveform whose fundamental may be sufficiently strong to sound like a human whistle. There are several classes of feedback as shown in Figure 1-6. There may be multiple characteristic speeds; one defining the speed of propagation of an instability and another defining the speed of the feedback (hydrodynamic or acoustical). There may be multiple characteristic lengths. One may contribute to the sound power while one or more may define each of several feedback paths.

1.4.2.1 Class I

The feedback is essentially incompressible; the speed of sound, although finite, is sufficiently large that it can be considered infinite. This action may be called near field or hydrodynamic feedback. There are a number Class I devices. A vibrating stick in a water stream, or a waving flag, is clearly due to hydrodynamic feedback. The Aeolian tone is another; it occurs in many applications, such as the telephone wire, the Aeolian harp, tree limbs, and even in flow meters [4.10.1]. A rarer version is the vortex whistle where a swirling flow exiting a tube results in a well defined oscillation due to hydrodynamic feedback [4.10.4]. The nature of these sources is discussed in the appropriate chapters.

1.4.2.2 Class II

The feedback is compressible and depends on the speed of sound. This may be called intermediate field, quasi-compressible feedback. A well known example is the edge tone where the interruption of the oscillating flow by a fixed surface generates a sound that disturbs the orifice flow [4.10.5]. A rarer example is the unstable supersonic jet. The nature of these sources is discussed in the appropriate chapters.

1.4.2.3 Class III

The feedback is compressible and depends on the speed of sound. This may be called far field or acoustic feedback. Examples are the hole tone [3.12.1] and ring tone [4.10.3] whose frequencies of oscillation can be determined by a distant reflecting surface. It can also be determined by a resonant structure such as in a musical instrument.
It should be noted that these classifications are more descriptive than rigid. The hole tone, for example can be either a Class II or III source.

**Fig. 1-6.** Category II classes of feedback that create flow generated pure tones or narrow band sound.

### 1.5 Flow Instability

Many sound sources are the result of unstable fluid motion, e.g., airfoils. Very slow flows are known to be laminar and are controlled by viscous forces; they are stable and if steady, create no sound. At

**Fig. 1-7.** Areas of flow stability and instability in a jet. **Fig. 1-8.** Unstable jet.
some speed, the laminar flow becomes unstable and amplifies small disturbances until the flow becomes chaotic (turbulent). The small disturbances can be temporal (hydrodynamic or sound) or spatial (surface roughness). There are small disturbances in every environment, so the transition is determined by the gain characteristics of the fluid which in turn is determined by the characteristics of the flow field. Two recent works on the hydrodynamic instability of flows [14, 15] discuss the theoretical framework for that process. Note that the use of “hydrodynamic” here is restricted to the instability, not necessarily to the disturbance or the consequences of the instability. One important source of instability is the presence of a velocity gradient or shear layer with an inflection point. Some successful theoretical studies of instability have concerned shear layers of infinite extent. The instability occurs simultaneously everywhere along the layer. Unfortunately, the interest here is in flows whose character changes in space (a jet or boundary layer). Thus, instability is a function of space as well as time, so any theory would apply only locally.

The original experiments of Osborne Reynolds (1842-1912) showed that instability was determined by the ratio of inertial to viscous forces, now known as the Reynolds number \( \text{A.2.2.4} \). The Reynolds number at which instability occurs depends on the frequency spectrum and magnitude of the disturbance. An example is shown in Figure 1-7 where a low speed rectangular water jet with a high aspect ratio was displaced laterally with a known amplitude and frequency. The value of \( D \) in the figure represents the ratio of the lateral disturbance displacement to the nozzle width; the disturbances were minute. The disturbance frequency was characterized by use of the Strouhal number \( \text{A.2.2.1} \). A photograph of the jet in the unstable region is shown in Figure 1-8. The spatial development of the instability is clear. Experiments with air jets produce very similar results. Flow instability plays an integral part in the sound from both Category I and II fluid sources.

### 1.6 Important Frequency and Amplitude Ranges

Too often the redesign of a product is made without taking into account the listener; the main aim, generally, is reduction of level. Although this monograph concerns the source primarily, successful changes to any product should include paying attention to the characteristics of the listener. In many cases, use of A-Weighting is only paying lip-service to the listener.

The level of sound experienced by a listener can range from less than 0 dB (2.9x10\(^{-9}\) lb/in\(^2\)) to more than 180 dB (2.9 lb/in\(^2\)). The frequency can range from well into the infrasonic to ultrasonic. Most concern about sound generation is contained within the range of human hearing which is more limited. For a normal hearing person, the frequency range is approximately from 20 to 20 kHz and the level range from 0 dB to about 140 dB where damage to the ear commences. Unfortunately, the level-frequency envelope is not rectangular.

The contours in Figure 1-9 are those of equal loudness in response to pure tones. This graph is particularly relevant to Category II sound sources in which there is a dominant frequency. The phon is a unit of perceived loudness; it compensates for the nonlinear nature of human hearing. The phon value of each curve is displayed above it while the value of the vertical axis is the actual sound pressure level. Phons are defined in ISO 226:2003. The scale was developed by S.S. Stevens (1906–1973) who also founded Harvard's Psycho-Acoustic Laboratory. MAF stands for the “minimum audible frequency”. The insensitivity of the ear at low frequencies is well known. What is not commonly appreciated is the extreme sensitivity in
the 2000 to 6000 Hz range. For example, reducing the frequency of a 4000 Hz Category II source to 1000 Hz, without change of level, provides an immediate 8 dB subjective level reduction. Use of these curves is deferred to {6.5}.

The contours in Figure 1-10 are those of equal noisiness in response to broad-band sound. This graph is particularly relevant to Category I sound sources that generate random spectra. The noy is a unit of perceived noisiness in each frequency band. When the noy is summed over all frequencies, it results in Noys (overall noisiness), which can be converted to PNdB (perceived noise in decibels). Examples of the use of the noisiness contours are given in {6.5}.

1.7 Key Points

There are only three basic sources of sound when a hypothetical surface is placed around a physical source so that the wave equation applies on it. The physical source can be any process that causes the surrounding fluid to transmit sound waves.

Sources have both a resistive and reactive component. It is important to recognize that sound fields (small amplitude resistive compressible motion) can be buried within a large scale incompressible reactive motion and so may not be recognized in the design of a product.

There are two categories of sound sources; the one in which feedback occurs (Category II) is generally critical to understand because of its narrow band frequency character.

When considering changes in the operation of a sound source, it is important to take into account the response of humans to those changes.

Fig. 1-9. Equal Loudness Contours for pure tones.

Fig. 1-10. Equal noisiness contours for broad-band sound.
Chapter 2
Waves and Plane Sources

Not all wave motion results in sound. In this chapter, waves and types of wave motion are discussed. The sound generated by plane surfaces is also discussed. Although a plane source is, in reality, a distribution of point sources, several important aspects of sound generation can be elucidated simply without the geometric complexity inherent in spherical coordinates.

2.1 Wave Motion

2.1.1 Definition of a Wave

A wave has variable displacement in either time or space. There are three categories of waves of interest:

1. those that are fixed relative to a medium, but move relative to an observer.
2. those that move relative to a medium but are fixed in time with respect to an observer.
3. those that move relative to a medium and move relative to an observer.

Examples of the first category are washboards on a road; the waves have spatial dependence but no time dependence. If a vehicle travels over the washboards, is sound necessarily created? Examples of the second category are “standing” waves, such as those found behind a rock in a water stream or those behind a mountain in an air stream (lenticular clouds). The waves have time dependence relative to the medium but only spatial dependence relative to a fixed observer. Do such waves make sound? No. If an observer travels through lenticular clouds on an aircraft, will sound necessarily be created? That is a different story; both the moving observer and a fixed observer on the ground will hear sound. The third category has two parts. The words “traveling waves” are used to describe waves that have both spatial and time dependence, and appear to move relative to an observer. The words “standing waves” are also used to describe waves traveling between two reflecting surfaces. Such waves have both spatial and time dependence, but appear not to be moving relative to an observer. Will the observer hear sound? Yes. The true traveling wave case is the one most commonly encountered.

Emphasis in this monograph is on the sound generation process. A fluid or solid must have elastic properties so that waves can exist in it. The waves will have a defined speed of propagation relative to the medium, a magnitude, and a direction of motion. Pressure is a measurable aspect of a wave in a compressible medium, so it is generally emphasized over the other aspects (mostly because it can be measured rather easily). If those waves generate a time dependent pressure fluctuation at an observer’s ear, it will be interpreted generally as “sound”, but it may not be. As will be shown in later chapters, there are time dependent pressure fluctuations that are not sound. Although both types of pressure fluctuations are discussed, the emphasis is on sound that propagates to distant locations.

Key Points: The correlation between the word “waves” and the word “sound” is not always one. Also, the correlation between pressure fluctuations that are heard and the sound generated by a source is not exact.
2.1.2 Motion of Plane Waves

Consider a wave moving in a positive (or negative) direction in space-time; the wave may have an (almost) arbitrary form as shown in the first of Eqs. 2.1. The wave shape at position zero and time zero has the same shape at distance $r$ or at the delayed (or advanced) time $t$. As long as the wave equation is valid {Appendix D} the time history at a remote location will be the same as that experienced near the source. For other geometries the time history is similar but the magnitudes are different. The wave form is applicable to both deterministic and random waves. The variable $c$ has the dimensions of $L/T$ and is interpreted to be a speed of propagation.

The second equation of Eqs. 2.1 may be used to express the amplitude $p$ as a function of time and space when the wave is sinusoidal (one frequency). The exponential function is a compact way of showing this dependence {Appendix B}. The most commonly used form is the term on the furthest right and will be the form used in later chapters. The wave number $k$ is based on the relationship between the frequency $f$ and wavelength $\lambda$ for a wave moving at speed $c$, as shown in the third expression. The wave number is not a number; it has the dimensions of inverse length: $L^{-1}$. Note that the wavelength is defined simply as a rotation of $2\pi$ (full circle).

Key Point: Waves with arbitrary shape that move in space-time without change of shape have a simple descriptor when they satisfy the wave equation. Mathematically, the function must not be discontinuous. Sound waves in air are not discontinuous. Ocean waves approaching a beach become discontinuous and curl over. For single frequency waves, specific relationships can be expressed.

2.1.3 The Elementary Plane Wave Equation

What equation of motion does the first of Eqs. 2.1 satisfy? Take the double space and time differentials for both the positive and negative functions; the result is shown in the first equation of Eqs. 2.2. The second equation results by equating the two double differentials, with an arbitrary constant set to zero. This must be the equation for the propagation of waves with unchanging shape. It is called the homogeneous wave equation because that constant is zero. If the term on the right is non-zero, it implies a forcing function (a sound source). The mathematical complexity inherent in such a function can be bypassed by studying only the space beyond the source; very little is lost in doing so. Since the variable $r$ is simply a
distance, a Cartesian coordinate system can be used to generalize the form of the equation. The first of Eqs. 2.3 below shows each of the Cartesian dimensions explicitly. The second uses vector notation. The upside down triangle is called the Laplacian operator, named after Pierre-Simon, marquis de Laplace (1749-1827) a French astronomer and mathematician. The square symbol encompasses the entire equation and is called the D’Alembertian or wave operator and is named after Jean le Rond D’Alembert (1717-1793). Note that if a fourth spatial dimension is defined as \( x_4 = \text{i}c t \), then the wave operator is merely a four dimensional Laplace equation. Did D’Alembert precede Einstein in defining space-time? The third equation is the wave equation in Cartesian tensor notation where \( i \) varies from 1 to 3 (by convention, the summation sign is implicit but omitted). The dates when these prodigious minds lived shows how early many of our modern concepts evolved.

It can be shown that the solution of the wave equation is correct for any continuous wave shape.

\[
\frac{\partial f^2}{\partial x^2} + \frac{\partial f^2}{\partial y^2} + \frac{\partial f^2}{\partial y^2} - \frac{1}{c^2} \frac{\partial f}{\partial t^2} = 0
\]

\[
\nabla^2 f = \frac{1}{c^2} \frac{\partial f}{\partial t^2} = 0
\]

\[
\frac{\partial f^2}{\partial x_i^2} - \frac{1}{c^2} \frac{\partial f}{\partial t^2} = 0
\]

**Key Points:** The wave equation can be expressed in several symbolic and geometric forms, and the solution for it can be expressed in general terms. Since the present subject is waves in air, the applicability of this equation to actual situations depends on a number of approximations to the equations of fluid motion \{D.2\}.

### 2.2 Wave Interactions

A number of variations to traveling waves occur and are described below. Trigonometric forms are used here in lieu of complex forms to show the physics more clearly. The amplitudes \( A \) should be considered as peak values.

#### 2.2.1 Traveling vs. Standing Single Frequency Waves

A traveling wave is the type discussed above. Standing waves that vary in time but do not appear to travel can be constructed from two equal, but opposing, traveling waves.

\[
f(r,t) = \sum_{m=1}^{n} A_m \left[ \cos(\omega_m t - k_m r) + \cos(\omega_m t + k_m r) \right] = 2 \sum_{m=1}^{n} A_m \sin \omega_m t \sin k_m r
\]

If \( \sin k_m r = \sin m\pi = 0 \), then \( \frac{\lambda}{2} = \frac{r}{m} \)

The time variations and spatial variations are separated by use of a standard trigonometric identity. Although the wave amplitude clearly varies with time, there are locations for each frequency in which the amplitude is zero, called the node points. This result is the exception to the standing waves described in \{2.1.1\}; it is actually composed of two traveling waves with a
specific phase relationship, typically caused by reflections from two facing surfaces. One interesting sidelight is that the sound intensity is zero (the two intensities cancel) although the sound pressure is finite, so it is possible to hear a sound with **zero intensity**!

**Key Points:** In the presence of a reflecting surface, measurements of sound pressure may not be representative of the undisturbed sound field. A classic example is a person holding a sound level meter between themselves and the sound source. The measurement of sound pressure does not always imply the correct sound intensity and thus the sound power.

### 2.2.2 Amplitude Modulated Single Frequency Waves

AM radio uses amplitude modulated electrical signals and the same process applies to sound. Consider two signals, one the carrier and one the modulation (information) signal which is less than the carrier frequency. Without modulation ($\omega_m=0$) the output signal $P$ in Eq. 2.5 oscillates with amplitude $A_0$ at the **carrier frequency** ($\omega_c$). With modulation, the amplitude of $A$ varies with time. When standard trigonometric identities are used, three signals result, one at the carrier frequency with no modulation and two **sidebands** of half amplitude. The information is carried in the sidebands only. Note that since the two frequencies are arbitrary, they do not necessarily have a harmonic relationship. To carry speech information, the modulation frequency must be expanded to a frequency spectrum, so the two sideband lines shown in Figure 2-1 must be broadened to include a band of frequencies. If $A_1$ is small, the amplitude envelope of the signal will vary only slightly. When $A_1/2 = A_0$, the amplitude envelop will go to zero at the modulation frequency rate. If $A_1/2 > A_0$, the modulation envelop will cross over, creating additional zeros. These latter conditions are avoided in AM radio.

**Key Point:** The importance here is that if a machine sound is modulated in a periodic way, e.g., RPM changes, a frequency below the main frequency (**sub-harmonic**) is generated. This modulation might be attributed erroneously to non-linear motion, an additional source, or an additional path.

\[ P = A \cos(\omega_c t) \]
\[ A = A_0 + A_1 \cos(\omega_m t) \]
\[ P = A_0 \cos(\omega_c t) + A_1 \cos(\omega_m t) \cos(\omega_c t) \]
\[ P = A_0 \cos(\omega_c t) + \frac{A_1}{2} \left[ \cos[(\omega_c + \omega_m) t] + \cos[(\omega_c - \omega_m) t] \right] \]
2.2.3 Phase Modulated Single Frequency Waves

Phase modulation is less common. For this example single frequency modulated phase

\[ P = A_0 \cos(\omega_0 t + \theta) \]
\[ \theta = A_1 \cos(\omega_m t) \]
\[ P = A_0 \cos(\omega_0 t + A_1 \cos(\omega_m t)) \]
\[ P = A_0 \left[ \cos(\omega_0 t) \cos(A_1 \cos(\omega_m t)) - \sin(\omega_0 t) \sin(A_1 \cos(\omega_m t)) \right] \]

shift will be used. There is no need for a fixed phase as it merely shifts the initial value of P.

Case 1: \( A_1 << 1 \)
For this case there are only two significant side bands similar to amplitude modulation.

Case 2: \( A_1 \leq 1 \)
This case is more complex and can be handled by series expansions. The exact result is best presented in software, since the expansion terms get quite long. Calculations for \( A_1 = 1 \) yield the spectrum shown in Figure 2-2 which has a multiplicity of sidebands whose relative levels are

\[ \cos(A_1 \cos(\omega_m t)) \approx 1 \]
\[ \sin(A_1 \cos(\omega_m t)) \approx A_1 \cos(\omega_m t) \]
\[ P = A_0 \cos(\omega_0 t) - \frac{A_1}{2} \left[ \sin[(\omega_0 + \omega_m) t] + \sin[(\omega_0 - \omega_m) t] \right] \]

given in Table 2-1 below. They drop relatively sharply.

![Fig. 2-2 Phase modulation spectrum.](image)

<table>
<thead>
<tr>
<th>N</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>dB</td>
<td>0</td>
<td>-4.7</td>
<td>-16.3</td>
<td>-31.6</td>
<td>-49</td>
</tr>
</tbody>
</table>

**Table 2-1. Amplitude of harmonics for a frequency modulated wave.**
2.2.4 Frequency Modulated Single Frequency Waves

\[ P = A_0 \cos(\omega t) \]
\[ \omega = \omega_c + A_i \cos(\omega_m t) \]
\[ P = A_0 \cos((\omega_c + A_i \cos \omega_m t) t) \]
\[ P = A_0 [\cos(A_i \sin(\omega_m t)) \cos(\omega_t t) - \sin(A_i \sin(\omega_m t)) \sin(\omega_t t)] \]
\[ P_n = J_{2n}(A_i) \left[ \cos((\omega_c + 2n\omega_m) t) + \cos((\omega_c - 2n\omega_m) t) \right] \]

FM radio uses frequency modulated electrical signals and the same concept applies to sound. Consider two signals, one the carrier and one the modulation (information) signal which is less than the carrier frequency.

To expand these functions further requires the use of Bessel function of the first order. The latter equation is representative of the nth sideband whose amplitude is expressed with the Bessel function which is oscillatory itself, so the sidebands do not decay monotonically as with the earlier types of modulations.

2.2.5 Beats

Beats create a form of modulation but are created by two signals of slightly different frequency, not the modulation of one signal. The sound from small twin engine propeller planes is an example of beats where the synchronizer attempts to match RPM, but never perfectly.

The two frequencies are subscripted \( c \) and \( m \) as in the previous sections. If they are greatly different, no beat is detected, but if they differ by small amounts, beats are heard. For this case we define \( \omega_d \) which is the difference between the frequencies. If that difference is small

\[ P = A_0 \cos(\omega_d t) + \cos(\omega_m t) \]
\[ P = 2A_0 \cos \left( \frac{\omega_c - \omega_m}{2} t \right) \cos \left( \frac{\omega_c + \omega_m}{2} t \right) \]
\[ P = 2A_0 \cos \left( \frac{\omega_d}{2} t \right) \cos \left( \omega_c - \frac{\omega_d}{2} t \right) \]

the additive factor in the second term of the third equation in Eqs. 2.8 may be neglected and we now have a situation similar to amplitude modulation where the modulation frequency is now the difference frequency.

**Key Points:** Apparent modulation of one sound source may be caused by two nearly synchronous sources. The identification is that the modulation frequency is considerably below that of the main source.
2.2.6 Finite Amplitude Waves

The speed of sound $c$ is based on isentropic conditions and for a perfect gas, i.e., small amplitudes $\{D.3\}$. A simple approach for larger amplitudes is as follows.

$$c = c_m + u = \sqrt{\gamma R(T_0 + T)} + u$$

$$c_m = \sqrt{\gamma R T_0 \left(1 + \frac{T}{T_0}\right)} = c_0 \left(1 + \frac{T}{2T_0} - \ldots \right) \approx c_0 \left(1 + \frac{\gamma - 1}{2} s \right)$$

$$u = c_0 s$$

$$c \approx c_0 \left(1 + \frac{\gamma + 1}{2} s \right)$$

The wave speed is the sum of the propagating wave in the medium plus the medium motion. $T$ is the temperature fluctuation and the square root expansion yields a first order expression in terms of condensation (See Eq. 2.10). When the medium motion is added the final relationship is as shown. At the peak ($s$ is positive) the wave travels faster than the trough ($s$ is negative). As a result the wave front steepens. The condensation $s$ is quite small at normal level sounds. Noticeable steepening occurs around levels of 170 dB, a pressure of around 1 psi, so use of the small amplitude sound speed is warranted for lesser levels. With time, the wave front in air steepens to a shock wave. When flow exits from a nozzle at 1.893 times (or more) the atmospheric pressure, the flow is critical and the flow goes supersonic $\{4.10.6\}$ and shock waves are formed. If this were an oscillatory pressure, it would equate to a sound level of 200 dB. Similarly, for aircraft moving faster than the speed of sound, the wavelets created steepen rapidly to create shock waves.

**Key Point:** The amplitude of most sound waves is sufficiently small that the wave shape remains constant so the spectrum contour of the sound does not necessarily change with distance.

2.3 Plane Wave Sources

The case of a large flat rigid surface that oscillates perpendicular to itself is given here. Motion in the plane of the surface is deferred until $\{2.3.7.1\}$ where viscosity plays a role not included in the standard wave equation (Eq. 2.3).

2.3.1 The Equations

The equations for the physical variables associated with waves created by the perpendicular motion of a plane surface in a normal fluid are given in Eqs. 2.10. The subscript zero refers to static values. See the development in $\{A.3\}$ for details of these variables. We see that by use of a solution for velocity potential, the various physical variables can be determined easily. Since interest is in sound generation, only the outgoing waves are discussed. Note that
the pressure \( p \), density \( \rho \), and temperature \( T \) variables are the changes from the non-zero static

\[
\frac{\partial^2 \phi}{\partial x^2} - i \frac{c_0^2}{\partial t^2} \phi = 0
\]

\[
c_0^2 = \frac{\gamma p_0}{\rho_0} = \gamma RT_0
\]

\[
p(x,t) = \rho_0 \frac{\partial \phi}{\partial t}
\]

\[
u(x,t) = \frac{\partial \phi}{\partial x}
\]

\[
s(x,t) = \frac{1}{c_0^2} \frac{\partial \phi}{\partial t} = \frac{\rho - \rho_0}{\rho_0}
\]

\[
T(x,t) = \frac{1}{R} \frac{\gamma - 1}{\gamma} \frac{\partial \phi}{\partial t}
\]

values while the motion variables are changes from zero values.

### 2.3.2 Force Generated Source

Consider the case of a surface with area \( A \) that is large with respect to a wavelength and having a sinusoidal force \( F \) applied to it. The magnitude of the force per unit area would be \( p_1 \).

The velocity potential is first developed from the pressure and then the other variables are

\[
p(x,t) = p_1 e^{i(\omega t - kx)}
\]

\[
\phi(x,t) = \frac{p_1}{i \omega \rho_0} e^{i(\omega t - kx)}
\]

\[
u(x,t) = \frac{p_1}{Z_0} e^{i(\omega t - kx)}
\]

\[
s(x,t) = \frac{p_1}{Z_0 c_0} e^{i(\omega t - kx)}
\]

\[
T(x,t) = \frac{1}{R} \frac{\gamma - 1}{\gamma} \frac{p_1}{\rho_0} e^{i(\omega t - kx)}
\]

derived from it, all are expressed in terms of surface pressure in Eqs. 2.11.

First note that all the properties vary with the applied force, but only the displacement and acceleration vary with frequency. For an applied force of 0.01 pound over an area of one square foot we get

<table>
<thead>
<tr>
<th>Frequency, Hz</th>
<th>Pressure Change, lb/in</th>
<th>Acceleration, ft/sec²</th>
<th>Velocity, ft/sec</th>
<th>Displacement, Inches</th>
<th>Density change, lb/ft³</th>
<th>Temperature change, °F</th>
<th>L_p, dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>0.00000694</td>
<td>12.16</td>
<td>0.00387</td>
<td>0.0000148</td>
<td>0.000000257</td>
<td>0.000715</td>
<td>87.6</td>
</tr>
<tr>
<td>2000</td>
<td>0.00000694</td>
<td>48.6</td>
<td>0.00387</td>
<td>0.00000370</td>
<td>0.000000257</td>
<td>0.000715</td>
<td>87.6</td>
</tr>
</tbody>
</table>

*Table 2-2. Value of physical variables at two frequencies for a constant force plane wave.*
Note that the density and temperature are in phase, so the compression of the medium results in a temperature rise. This is based on the process being adiabatic, and is reasonable since it is unlikely that heat transfer can occur for such a fast process.

The Sound Pressure Level is shown below. The magnitude of $p_1$ should be interpreted as the root-mean-square (rms) value, not the peak value \{B.4\}.

$$L_p = 10\log_{10}\left[\frac{p_1^2}{p_R^2}\right] = 20\log_{10}\left[\frac{p_1}{p_R}\right]$$

**Key Point:** It does not take much force to generate a high sound pressure level. Note that the pressure and velocity are in phase, so all the motion is resistive, i.e., all the work applied to the medium at the boundary is transferred to the medium as sound.

### 2.3.3 Displacement Generated Source

Consider the same situation as in the previous section except that the sinusoidal force is replaced by a sinusoidal displacement of fixed amplitude.

$$\varepsilon(x,t) = \varepsilon_0 e^{i(\omega t-kx)}$$
$$\phi(x,t) = \varepsilon_0 c_0 e^{i(\omega t-kx)}$$
$$p(x,t) = i\omega \varepsilon_0 Z_0 e^{i(\omega t-kx)}$$
$$u(x,t) = i\omega \varepsilon_0 e^{i(\omega t-kx)}$$
$$s(x,t) = ik \varepsilon_0 e^{i(\omega t-kx)}$$
$$T(x,t) = i\omega \varepsilon_0 \frac{1}{R} \gamma - 1 e^{i(\omega t-kx)}$$

$$L_p = 20\log_{10}\left[\frac{\omega \varepsilon_0 Z_0}{p_R}\right]$$

In this case many of the variables are now dependent on frequency unlike the previous case. There is a displacement problem; the velocity of the surface must match that of the fluid at the boundary $\xi$ which is not zero. A series expansion yields the relationship on the left.

\[
u(t) = U e^{i(\omega t-kx)} \approx U e^{i(\omega t-kx)} \left[ 1 - ik \varepsilon + \frac{k^2 \varepsilon^2}{2!} - \ldots \right]
\]

Looking at the form of the velocity in Eqs. 2.12, we see that the magnitude of the succeeding terms in the expansion are in powers of the local Mach number (as seen in the expression on the right). Thus the Mach number of the wall motion must be small in order to permit the approximation that the interface is at \(x=0\). Another, and more satisfying, view is that the displacement must be small compared to the wavelength. Once again the pressure, velocity, density and temperature are all in phase, so all motion dissipates the power by radiation and nothing is stored temporarily. Note that if the area of the plane is finite, say $A$, it can be considered to be a volumetric flow source $uA$. 

2-9
Key Point: The Mach number of a vibrating surface must be small if the frequency spectrum of the sound field is to be the same as that of the source.

2.3.4 Thermally Generated Source

Normally, one is not concerned with sinusoidal oscillations of temperature as a source of sound; the magnitude of temperature changes associated with normal sound pressure levels are small. A large and abrupt temperature change can result in significant levels. In this case, the temperature is given; the resultant relations are shown on the right. The Sound Pressure Level is

\[ L_p = 10 \log_{10} \left[ \frac{\alpha^2 \rho_0^2 T_i^2}{p_R^2} \right] = 20 \log_{10} \left[ \frac{\alpha p_i T_i}{p_R} \right] \]  \hspace{1cm} (2.13)

shown on the right. The Sound Pressure Level is \( T_i \) should be interpreted as the rms value, not the peak value \( B.3 \).

An unusual situation is the periodic, diurnal temperature variation through the wall of a building. Is sound created by this process? According to the equations, the answer would be yes, but the frequency would be at 1.1 x 10^{-5} Hz, well out of the human hearing envelope! This is an example of working beyond the bounds of validity. The equation of state \( D.3 \) requires small amplitude and rapid temperature changes which these are not, so no is the correct answer. A more realistic application is a current pulse through a thin metallic plate of finite resistance. The power absorbed can be considered a step function, resulting in a rapid temperature rise and thus sound.

Key Point: Temperature changes as sound generating mechanisms in plane surfaces are rare. There are many cases of thermally generated sources in other geometries, such as lightning, corona noise, and combustion \{3.11.1\} and consideration should be given to it.

2.3.5 Acoustic Loading of a Vibrating Surface

When vibration of a plate is considered, acoustical aspects are seldom addressed. Is acoustical loading a significant factor? Consider the simple mass, spring, resistor arrangement shown in Figure 2-4. Many books show the methods of analysis; the relevant equations are shown below.

\[ M \ddot{\xi} + (R + Z_0 A) \dot{\xi} + K \xi = 0 \]
\[ \xi = Be^{\alpha t} \]
\[ \omega_n^2 = \frac{K}{M} \xi = \frac{(R + Z_0 A)}{2M \omega_n} \]
\[ \omega_d = \omega_n \sqrt{1 - \xi^2}, \alpha = \xi \omega_n \pm i \omega_d \]
\[ \xi = Be^{-\xi \omega_d t} \cos(\omega_d t + \theta) \]  \hspace{1cm} (2.14)

---

Fig. 2-4 Mass-Spring
The symbol $\xi$ is the displacement, and $\omega_n$ is the undamped natural frequency. The damped natural frequency $\omega_d$ depends on both the internal damping $R$ and the radiation damping $Z_0A$. The acoustical loading appears in the resistance term; since it is per unit area the mass area needs to be included. If the damping factor is less than one, the last equation applies. What is the effect of acoustic loading on the resonance frequency and on the rate at which the amplitude decays? The geometry of the mass is quite important. If the mass is thick with a small surface area the acoustic loading is small. If the mass is a thin plate, the story is different. Figure 2-5 shows the amplitude decay as a function of time with, and without, acoustic damping. The data were: $M=17$ lb$_m$, $R=10$ lb$_f$-sec/ft, $K=30$ lb/in. The mass was 1.75 inches thick and 9.75 inches square. The damped natural frequency was 24.3 Hz. without acoustic loading and 23.6 Hz with it.

There are several other aspects related to control of sound emission from vibrating surfaces. Figure 2-6 shows the velocity response of a mechanical resonance system with damping as a function of frequency being driven by a constant force. Every text on mechanical vibration has equivalent graphs, but often attempts to reduce sound emission from such a system make poor use of it. The response on the right side of the graph is called the mass controlled region while that on the left side is called the stiffness controlled region. Between them is the damping controlled region. The key, of course, is to know what the natural (damped resonance) frequency is with respect to the frequency range of human hearing.

If the resonant frequency is low, it is beneficial to add limp, not stiff, mass to drive it to a lower frequency. Stiffness has been added for this situation in the mistaken assumption that it will reduce sound emission because it reduces displacement motion.

Fig. 2-5 Acoustic damping of a vibrating plate.

Fig. 2-6 Response of a resonant mechanical system to a constant force.
The role of damping in the graph is clear. It is highly beneficial to reduce resonance response, but only if that range of frequencies is relevant. Damping has been added to plates in which the resonant frequency is so low the only function of the material was to add a little mass and a lot of cost. The same is true in the stiffness controlled region. For plates with multiple resonances, damping is very beneficial.

What basic damping mechanisms are there? The parentheses show the proportionality of the damping forces.

- Fluid viscous \( (\xi) \)
- Fluid compressible pressure (sound) \( (\xi) \)
- Fluid laminar incompressible pressure \( (\xi) \)
- Fluid turbulent incompressible pressure \( (\xi^2) \)
- Solid Hysteresis \( (\xi) \)
- Solid Coulomb (Dry friction)

**Key Points:** Large radiating areas play a significant role in the sound radiated from a surface. Care must be taken when choosing the methods of velocity reduction on resonant structures.

### 2.3.7 Influence of Viscosity

The wave equation does not contain viscous terms. Does that have significant influence on results?

#### 2.3.7.1 Motion in the Plane of a Flat Surface

When a plane surface moves in the same direction as the plane, the fluid is not compressed and viscous forces control. Eq. 2.15 shows the fluid velocity \( v \) in the direction of the plane \( y \) as a function of the distance \( x \) perpendicular to the plane for a single frequency. The medium responds to the motion of the plane by propagating a damped shear wave perpendicular to the plane. The damping constant is proportional to the frequency and inversely to the viscosity.

\[
v(x, y, t) = v_0 e^{-a x} \cos(\omega t - ky)
\]

**Key Points:** Motion in the plane of a surface creates only transverse or shear waves and will not create sound. Exponentially damped shear waves are propagated into the medium. The characteristic speed is that of the surface motion. The characteristic length is the wavelength of the motion.
2.3.7.2 Motion Perpendicular to the Plane of a Flat Surface

When the surface moves perpendicular to itself with viscous terms included in the governing equations, the solution for a single frequency is as shown in Eqs. 2.16.

\[ \frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} \]

\[ u(x,t) = Ae^{i(\omega t - \xi x)} \]

(2.16)

\[ k = \frac{\omega}{c_0} \left[ 1 + \frac{i\nu \omega}{c_0^2} \right]^{\frac{1}{2}} \approx \frac{\omega}{c_0} + \frac{i\nu \omega^2}{2c_0^3} \]

\[ u(x,t) = Ae^{-\frac{\nu \omega^2 x}{c_0^2}} \cos \left[ \omega \left( t - \frac{x}{c_0} \right) \right] = Ae^{\frac{-4\pi^2 x}{Re_v \lambda}} \cos \left[ \omega \left( t - \frac{x}{c_0} \right) \right] \]

\[ Re_v = \frac{c_0 \lambda}{\nu} \]

The expansion of the wave number \( k \) is based on the second term being small. Viscous effects do attenuate sound. The attenuation increases rapidly with frequency. How much? The attenuation coefficient can be likened to an acoustical version of the Reynolds number. In air, for a frequency of 1000 Hz, the attenuation is \( 4.5 \times 10^6 \) foot. The neglect of viscous effects for most frequencies of sound in air is warranted.

**Key Points:** Motion perpendicular to a surface creates longitudinal (compression) waves and will create sound. Viscosity causes exponential decay of the sound amplitude, but the attenuation coefficient is sufficiently small that it can be neglected in the hearing range of humans.

2.3.8 Limitations of Plane Wave Analysis

Can unbounded plane waves actually exist? In the above analyses, the source is presumed to be an infinite plane, all parts of which are fully coherent. Although the model provides much insight, it is not physically realizable. The first clue, of course, is that the sound level does not decrease with distance! The closest approximation to the real world would be a finite plane surface whose dimensions are longer than several wavelengths. It must also be of significant depth in wavelengths to avoid communication with the back side. If the normal frequency range of hearing is considered, this would imply a large and very stiff structure. Consequently, the best approximation to a plane source is to be in the plane wave region, the triangular region shown in Figure 2-7. This limited region has some value in that useful information about local motion of a complicated source can be obtained.

Of course bounded plane waves can exist, such as the propagation of sound in a duct. However, when this geometry exists, the presence of boundaries allows for other modes of
propagation, called *cross-modes*. Depending on the shape of the boundary (e.g., a rectangular or cylindrical duct), these modes can take on several forms and it is necessary to separate them from the plane wave mode.

### 2.4 Body Wave Types

It is possible to class mechanical waves into two types; body and surface. Both types have the ability to result in sound.

#### 2.4.1 Longitudinal (Compression) Waves

These waves result in the compression of the medium (density) changes. The main characteristic is that the material motion (either solid or fluid) is in the same direction as that of the wave. The wave is associated with pressure forces that create the motion. Each of the basic sound sources creates this type of wave to propagate the source energy to distant locations (the sound field). The large arrows in Figure 2-8 are the directions of fluid motion.

#### 2.4.2 Transverse (Shear) Waves

These waves result in lateral oscillatory motion of the fluid. The main characteristic is that the material motion is in the plane and at right angles to the direction of the wave. It is associated with the viscous forces that create the motion. Each of the basic sound sources creates this type of wave but in most cases the magnitude of the motion is sufficiently small that it is neglected (2.3.7).

### 2.5 Surface Wave Types

Surface waves can occur on elastic materials of any depth, such as on the surface of the earth, or on plates. Both have the potential to create sound in the adjacent medium.

#### 2.5.1 Water Waves

These waves operate in liquids of finite but often great depth. They are waves that move along the surface and interact with the surrounding medium. Although they decay in magnitude with depth; the fluid motion is circular in a plane whose axis is at right angles to the direction wave travel, as seen in the first of Figure 2-9. They have a component of motion vertical to the surface.

#### 2.5.2 Rayleigh Waves

These waves operate in materials of great depth. They are waves that move along the surface and interact
with the surrounding medium. Although they decay in magnitude with depth, the material motion is retrograde elliptical in a plane whose axis is at right angles to the direction of wave travel, as seen in the second of Figure 2-9. They have a component of motion vertical to the surface also. The waves are named after J.W. Strutt (Baron Rayleigh 1842-1919).

### 2.5.3 Love Waves

These waves also operate in materials of great depth, and are waves that move along the surface and interact with the surrounding medium. Although they decay in magnitude with depth, material motion is lateral to the direction of the wave, as seen in the third of Figure 2-9. They are propagating shear waves and do *not* have a vertical component of motion as might be suggested by the simple diagram in the figure. The waves are named after A. E. H. Love (1863-1934).

### 2.5.4 Lamb Waves

These waves operate in materials of finite thickness. They are essentially internal longitudinal waves that cause the material to bulge and thin due to the Poisson effect causing surface waves that interact with the surrounding medium. The theory was developed by Horace Lamb (1849-1934).

### 2.5.5 Sound from Surface Waves

In each of the above cases, there is sinusoidal motion of the surface which is coupled to the surrounding medium. Does such motion create sound? There are many texts that address surface wave motion and the mathematics can get complex so only a general discussion will be given here. There are two cases: when the waves are traveling and when they are standing.

*Standing waves* occur when the surface is of finite length, such as a metal plate or supported membrane. All such motion has a prescribed frequency and node points between which the motion is finite. *Sound is created.* A simple model of standing waves is given in (3.11.4) and in the *SoundSource* program.

*Traveling waves* couple to the medium in different ways, depending on the surface wave speed. Morse and Ingard give a detailed mathematical analysis of this situation [1, p624]. When the surface wave speed \( C_p \) is less than that of sound in the medium \( C_0 \), the medium motion is reactive and the periodic motion moves with the surface wave but decays exponentially in the direction perpendicular to the surface. *No sound is created.* When the wave speed is greater than the sound speed, *sound is created.* The waves move outward at the Mach angle \( \cos \theta = \frac{c_0}{c_p} \). As the surface wave speed just passes sound speed the waves move along the surface as if a plane source was on one end of the surface. If the surface wave speed is much greater than the sound speed the waves

---

Fig. 2-10. Medium motion due to a surface wave.
speed, the waves move almost perpendicular to the surface, much like a plane surface moving perpendicular to itself.

**Key Points:** Standing surface waves will always create sound. Travelling surface waves can create sound when the surface wave speed exceeds that of sound in the medium. The radiation has a specific direction based on the ratio of the two speeds. The characteristic speed is that of the surface wave.
Chapter 3  
Monopole Sources  

3.1 The Mathematical Model  

First, mathematical models of the monopole are developed to provide as much useful information as possible. Then real applications are discussed and analyzed with use of the mathematical model and the scaling rules.  
The basic wave equation for the monopole has spherical symmetry.  

\[
\frac{\partial}{\partial \theta} = 0, \frac{\partial}{\partial \psi} = 0
\]  
\[
\frac{\partial^2}{\partial r^2}(r \phi) - \frac{1}{c_0^2} \frac{\partial}{\partial t^2}(r \phi) = 0
\]  
\[
\frac{\partial}{\partial r^2}(r \Phi) + k^2 (r \Phi) = 0
\]  

\(\phi(r, t)\) and \(\Phi(r, \omega)\) are the velocity potentials. The general solution in the time domain is that any continuous but arbitrary function satisfies the wave equation and is propagated radially outward in time without change of form as shown in the first of Eqs. 3.2. The single frequency solution is given in the second equation.  

\[
\phi(r, t) = \frac{1}{4\pi r} f \left( t - \frac{r}{c_0} \right)
\]  
\[
\phi(r, t) = \frac{Q_m}{4\pi r} e^{i(\omega t - kr)}
\]  
The main variables are  

\[
p(r, t) = \rho_0 \frac{\partial \phi}{\partial t} \quad u_r(r, t) = -\frac{\partial \phi}{\partial r} \quad u_\theta = u_\psi = 0
\]  

3.1.1 Physical Interpretation of the Source  

To properly model the monopole it is necessary to look at the equations as representative of a physical process. Consider the source as an arbitrary function of time; by use of the Helmholtz equation we get  

\[
\Phi(r, \omega) = \frac{q_m(\omega)}{4\pi r} e^{-ikr}
\]  
\[
q_m(\omega) = \int_{-\infty}^{\infty} Q_m(t) e^{-i\omega t} dt
\]  
\[
Q_m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q_m(\omega) e^{i\omega t} d\omega
\]
The source spectrum is defined by the Fourier transform pair. By making use of the shifting theorem the time dependent value of the velocity potential becomes

$$
\phi(r,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(r,\omega) e^{i\omega t} d\omega = \frac{1}{4\pi r} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} q_m(\omega) e^{i\omega (t-r/c_0)} d\omega \right] = \frac{Q_m \left( t - \frac{r}{c_0} \right)}{4\pi r}
$$

Although this result seems trivial, it leads to some interesting results. The physical variables are

$$
p(r,t) = \rho_0 \frac{\partial Q_m \left( t - \frac{r}{c_0} \right)}{\partial t}
$$

$$
u_r(r,t) = \frac{1}{4\pi rc_0} \left[ \frac{\partial Q_m \left( t - \frac{r}{c} \right)}{\partial t} + c_0 Q_m \left( t - \frac{r}{c_0} \right) \right]
$$

$$
I_m(r) = \bar{p} u_r = \frac{Z_0}{16\pi^2 r^2 c_0^2} \left[ \frac{\partial Q_m}{\partial t} \frac{\partial Q_m}{\partial t} + Q_m \frac{\partial Q_m}{\partial t} \right]
$$

The time retardation factor was left out of the intensity expression for simplicity. The bars represent the time average. The second term in the velocity and intensity expressions represents the reactive motion near the source. By putting the mean fluid density into the expression, the resistive part of the radial intensity becomes

$$
I_m(r) = \frac{Z_0}{16\pi^2 r^2 c_0^2} \left( \frac{\partial Q_m}{\partial t} \right)^2 = \frac{1}{16\pi^2 r^2 Z_0} \left( \frac{\partial M}{\partial t} \right)^2
$$

(3.4)

$M$ is the mass flow rate. The sound power can be expressed simply as

$$
W = \frac{1}{4\pi Z_0} \left( \frac{\partial M}{\partial t} \right)^2
$$

(3.5)

The sound from a monopole is created by the mean square of the time rate of change of the mass flow rate.
3.2 The Single Frequency Point Monopole

3.2.1 Physical Variables

A solution of the wave equation for a single frequency can be expressed in terms of the velocity potential \( \phi \) as shown in the first of Eqs. 3.6. There is an incoming wave solution also, but that has been discarded since we are interested in radiating sources. The quantity \( Q_m \) is called the source strength and a check of units will show that it has the dimensions \( L^3/T \), a volumetric flow rate. Several of the physical variables are also shown below in terms of the velocity potential.

\[
\phi (r,t) = \frac{Q_m}{4\pi r} e^{(i\omega t - kr)} \quad \xi_r (r,t) = \frac{(kr - i)}{krc_0} \phi
\]

\[
p (r,t) = \frac{ikZ_0Q_m}{4\pi r} e^{(i\omega t - kr)} = ikZ_0 \phi \quad s (r,t) = \frac{ik}{c_0} \phi
\]

\[
u_r (r,t) = \frac{(1 + ikr)}{r^2} \frac{Q_m}{4\pi} e^{(i\omega t - kr)} = \frac{(1 + ikr)}{r} \phi \quad T (r,t) = \frac{i\omega (\gamma - 1)}{R} \phi
\]

\[
a_r (r,t) = \frac{ikr (1 + ikr)}{c_0} \phi
\]

The scalar quantities, pressure \( p \), temperature \( T \), and condensation \( s \) remain in phase as expected. The radial acceleration \( a_r \), velocity \( u_r \), and displacement \( \xi_r \) are at quadrature with each other. Depending on radius, the radial velocity can be either in phase with the pressure or at quadrature with it. This is related to the compressible-incompressible flow regions around the source. The close-in region is called the near sound field and the other the far sound field.

It is possible to define local Strouhal and Mach numbers as shown on the right. The characteristic speed is the “particle” speed \( St = \frac{fr}{|u_r|}, Ma = \frac{|u_r|}{c_0} \) and the characteristic length radial distance. Note that the local Strouhal number increases with radius and the local Mach number decreases with radius, i.e., the validity of the wave equation improves with radius \{D.2\}.

3.2.2 Near Field

Close to the origin, the radial acceleration is nearly in phase with the pressure; the pressure is used primarily to accelerate the fluid; the motion is reactive and essentially incompressible. Since the velocity is at quadrature with the pressure, very little work is being done.

\[
p (r,t) = \frac{ikZ_0Q_m}{4\pi r} e^{(i\omega t - kr)}
\]

\[
u_r (r,t) = \frac{Q_m}{4\pi r^2} e^{(i\omega t - kr)}
\]

\[
a_r (r,t) = \frac{ikQ_m}{4\pi r^2 c_0} e^{(i\omega t - kr)}
\]
done on the medium. The sound field is a smaller part of the motion.

### 3.2.3 Far Field

Far from the origin the phase of the velocity shifts to be the same as the scalar variables. The reactive motion is minimized, and work is being done on the medium. The sound field is dominant.

\[
p(r,t) = \frac{ikZ_0Q_m}{4\pi r} e^{i(\omega t-kr)}
\]

\[
u_r(r,t) = \frac{ikQ_m}{4\pi r} e^{i(\omega t-kr)}
\]

\[
a_r(r,t) = -\frac{k^2Q_m}{4\pi rc_0} e^{i(wt-kr)}
\]

### 3.2.4 Radiation Impedance

The impedance to radial motion has both reactive and resistive components. Close to the source the reactive (incompressible) components dominate. Further out the resistive (sound field) components dominate. The impedance can be expressed in several ways as shown below.

\[
Z_r = \frac{p}{u_r} = Z_0 \frac{ikr}{1+ikr} = Z_0 \frac{kr}{\sqrt{1+k^2r^2}} e^{i\left(\frac{\pi}{2}-\alpha\right)} = R_m + iX_m
\]

\[
\tan \alpha = kr
\]

\[
R_m = Z_0 \frac{k^2r^2}{1+k^2r^2}, X_m = Z_0 \frac{kr}{1+k^2r^2}
\]

### 3.2.5. Sound Intensity

The power flow per unit area, or intensity, is purely radial. It is

\[
I_r(r) = \mathbf{p} \cdot \mathbf{u}_r = \frac{k^2Z_0Q_m^2}{16\pi^2 r^2} \left[1 + \frac{i}{kr}\right]
\]

At large kr (not r alone) the pressure and velocity are in phase as in the plane wave case; the sound intensity is the real part of the expression. For a given volumetric flow rate, the intensity increases with the square of the frequency, decreases with the square of the distance, and increases with the impedance of the medium.

Looking at the reactive part of the intensity, we see that it decreases with the cube of the distance, suggesting that it is more like an incompressible flow field. It is reasonable to set the near/far field boundary at \(kr = 10\). The radius of the boundary decreases with increasing frequency, suggesting a smaller reactive volume at high frequencies.

**Key Points:** In some documents, the denominator has 8 in Eq. 3.7, in lieu of 16, because the value of \(Q_m\) was described as the peak amplitude while here it is described as the r.m.s.
amplitude, the difference of the squares being two \( \{A.2\} \). This also applies to the sound power equation below \( \{3.2.7\} \).

### 3.2.6 Estimates of Sound Intensity

Direct intensity measurements and their integral to estimate sound power require specialized equipment. How well does a sound pressure measurement approximate to intensity? The relationship for this is

\[
EST[I_r] = \frac{p \cdot p^*}{Z_0} = \frac{k^2Z_0Q_m^2}{16\pi^2 r^2}\quad (3.8)
\]

A mean square pressure measurement will represent the intensity correctly, despite the presence of some incompressible flow (provided, of course, that the source is in fact a monopole). The sound pressure level and source strength are

\[
L_p = 10\log_{10} \left[ \frac{k^2Z_0Q_m^2}{16\pi^2 r^2 p_R^2} \right]
\]

\[
Q_m = \frac{2rp_Re_c}{fZ_0}\left( \frac{L_p}{20} \right)
\]

The following relationships result:

- 6 dB/octave increase per doubling of frequency
- 6 dB increase per doubling of source strength
- 6 dB decrease per doubling of distance

Note that the above results imply that the sound level will increase indefinitely with frequency - pure nonsense. At some point, the wavelength will be on the order of the actual source size. This is taken into account in \( \{3.3\} \) below. In many practical cases, this is not an issue, but it is important to be aware of this limitation \( \{6.1.4\} \).

### 3.2.7 The Sound Power

The power is the integral of intensity over the sphere

\[
W_m = \frac{\mu}{2\pi} \int_0^\pi \int_0^{2\pi} I_r(r) r^2 \sin \theta d\theta d\psi = 4\pi r^2 I_r(r) = \frac{k^2Z_0Q_m^2}{4\pi}\quad (3.9)
\]

The acoustic power is the real part of the expression. Again, we see that the wave equation cannot be correct for a truly point source since the reactive power requirements tend toward infinity as \( r \) approaches zero \( \{6.1.1\} \).
3.2.8 Dimensional Analysis

By introducing dimensionless factors into the sound power equation, it can be presented in a different form by adding the characteristic length \( L \) and speed \( U \) (undefined at this point).

\[
W_m = \frac{\pi P_0}{c_0} \hat{Q}^2 S t^2 U^4 L^2 \\
\hat{W}_m = \pi \hat{Q}^2 S t^2 M
\]

(3.10)

Definitions of the dimensionless variables are in \( \{A.2.2\} \). This equation applies for a single frequency point monopole. It can also apply for a single frequency finite monopole whose radius is much less than a wavelength \( \{3.3.3\} \), i.e., the Helmholtz number is much less than one \( \{A.2.2.2\} \).

3.3 The Single Frequency Finite Monopole (Pulsating Sphere)

3.3.1 The Physical Variables

All of 3.2 is useful but still fiction; real sources have finite dimensions. Note that for the point monopole the volumetric flow rate remains constant despite the zero size of the source. Worse, the sound pressure increases as the square of the frequency ad infinitum. As the source is approached, the radiation impedance goes to zero. It is clear that the approximations made to develop the wave equation must of necessity be invalid for small values of the radius \( \{6.1.1\} \). Consider a situation closer to reality, a monopole of finite size; a pulsating sphere of radius \( a \). The boundary conditions on the surface of the sphere determine the resulting sound field. The radial velocity on the surface of the sphere is related to the volumetric flow rate as

\[
u_r(a,t) = u_a e^{i\omega t} \\
Q_m = 4\pi a^2 u_a
\]

(3.11)

Since the surface of the sphere and fluid immediately in contact with it must have the same motion, the Mach number of the velocity must be small \( \{2.3.3\} \). In the point source solution, the phase was chosen with respect to \( r=0 \) and now it must be chosen with respect to \( r=a \). The important variables become

\[
\phi(r,t) = \frac{1}{1 + ika} \frac{Q_m}{4\pi r} e^{i(a-k(r-a))} \\
p(r,t) = \frac{i\alpha}{1 + ika} \left( \frac{a}{r} \right) Z_0 u_a e^{i(a-k(r-a))} \\
u_r(r,t) = \frac{1 + ikr}{1 + ika} \left( \frac{a}{r} \right)^2 u_a e^{i(a-k(r-a))}
\]

(3.12)
3.3.2 Near Field

For \( kr \ll 1 \), the following expressions apply. The radial velocity grows as the square of the distance as the source is approached and is at quadrature with the pressure, suggesting the dominance of the incompressible flow field.

\[
p(r, t) = \frac{ika}{1 + ika} \left( \frac{a}{r} \right) Z_0 u_a e^{i(\omega t - kr)}
\]
\[
u_r(r, t) = \frac{1}{1 + ika} \left( \frac{a}{r} \right)^2 u_a e^{i(\omega t - kr)}
\]

3.3.3 Far Field

For \( kr \gg 1 \), the following expressions apply. The radial velocity and pressure are in phase and each decrease linearly with distance.

\[
p(r, t) = \frac{ika}{1 + ika} \left( \frac{a}{r} \right) Z_0 u_a e^{i(\omega t - kr)}
\]
\[
u_r(r, t) = \frac{ikr}{1 + ika} \left( \frac{a}{r} \right)^2 u_a e^{i(\omega t - kr)}
\]

3.3.4 Source Size

For a source that is large with respect to the radiated frequency (\( ka \gg 1 \)), the following expressions apply. This would be a very rare occasion. Note that frequency dependence is lost; the source is more like a plane source.

\[
p(r, t) = \left( \frac{a}{r} \right) Z_0 u_a e^{i(\omega t - kr)}
\]
\[
u_r(r, t) = \left( \frac{a}{r} \right) u_a e^{i(\omega t - kr)}
\]

3.3.5 Radiation Impedance

The radial impedance at the surface of a finite sized sphere is

\[
Z_a = Z_0 \frac{ika}{1 + ika} = R_a + iX_a
\]

\[
R_a = Z_0 \frac{k^2 a^2}{1 + k^2 a^2}
\]

\[
X_a = Z_0 \frac{ka}{1 + k^2 a^2}
\]  \hspace{1cm} (3.13)

Fig. 3-1. Finite monopole impedance.
The radiation resistance increases with the square of the frequency as with the point monopole, but as \( ka \) approaches one, it becomes limited, bringing the results closer to reality. Both resistance and reactance are 3 dB down when the Helmholtz number is one \((ka=1)\). The \textit{acoustical} Strouhal number is 0.16, based on the characteristic length being the source radius \( a \), and the characteristic speed being \( c_0 \). At low frequencies, the resistive impedance increases at 6 dB/octave while the reactive impedance increases at 3 dB/octave. The hydrodynamic motion (fluid acceleration) dominates at low frequencies while at high frequencies the acoustic motion (fluid compression) dominates.

**Key Points:** For a given volumetric (mass) flow rate, sound output increases with frequency so frequency reduction (speed, RPM) will reduce output. The impedance applies to broadband sources, so the radiated spectrum at very high frequencies will not be the same as the initiating spectrum at the source.

### 3.3.6 Sound Intensity

The sound intensity is given in Eqs. 3.14. The reactive component decays with distance more rapidly than the resistive. Note again that for a source large with respect to a wavelength, there is no frequency dependence, the impedance maximizes at some point, as suggested in Figure 3-1.

\[
I_r(r,t) = u_a^2 Z_0 \left[ \left( \frac{a}{r} \right)^2 \frac{k^2 a^2}{1 + k^2 a^2} + i \left( \frac{a}{r} \right)^3 \frac{ka}{1 + k^2 a^2} \right]
\]

\[
I_r(r,t) = \frac{Z_0 Q_m^2}{16 \pi^2 \left( 1 + k^2 a^2 \right)^2} \left[ \frac{k^2}{r^2} + i \frac{k}{r^3} \right]
\]

### 3.3.7 Estimates of Sound Intensity

Direct intensity measurements and their integral to estimate sound power require specialized equipment. How well does a sound pressure measurement approximate to far field intensity? The relationship for this is

\[
EST[I_r] = \frac{p \cdot p^*}{Z_0} = u_a^2 Z_0 \left( \frac{a}{r} \right)^2 \frac{k^2 a^2}{1 + k^2 a^2}
\]

A mean square pressure measurement will represent the far field intensity correctly, despite the presence of some incompressible flow (provided, of course, that the source is in fact a monopole).
### 3.3.8 Sound Power

The sound power can be expressed in a simple way. For a given surface velocity, the sound power increases with the sphere’s surface area $A_s$, as expected.

$$W_m = 4\pi r^2 I_r = \frac{Q_m^2 k^2 Z_0}{4\pi \left(1 + k^2 a^2\right)} = 4\pi a^2 u_a^2 Z_0 \frac{k^2 a^2}{1 + k^2 a^2} = A_s u_a^2 R_m$$  \hspace{1cm} (3.16)

### 3.3.9 Dimensional Analysis

Eq. 3.16 is the same as for a point monopole (Eq. 3.10) when $ka \ll 1$. When $ka \gg 1$, the form of Eq. 3.11 must be altered as shown below. The speed dependence is reduced, the sound power is no longer dependent on frequency, and the characteristic length is now defined. This equation applies only for high frequencies, since the value of $ka$ implies wavelengths that are quite short with respect to the radius. It is likely that such a case occurs rarely. For broad-band sound, the change in speed dependence of the high frequency components will distort the spectrum; this is discussed in {6.1.4}.

For a finite monopole with a very large radius with respect to wavelength, the dependence on the steady speed changes from $U_4$ to $U_2$ as shown below

$$W_m = \frac{Z_0}{4\pi} \hat{Q}^2 U^- a^2$$

### 3.4 The Interaction of Two Single Frequency Finite Monopoles

#### 3.4.1 The Mathematical Model

Consider two monopoles of equal size separated by a distance $2h$. The velocity potential at the measurement point is the sum of each as shown in Eq. 3.17. The variable $\delta$ is an arbitrary phase angle between the sources. The measurement angle starts at the vertical.

$$\phi(r,t) = \frac{1}{1 + ika \frac{Q_{m1}}{4\pi r_1}} e^{i(\omega t - k(r_1 - a))} + \frac{1}{1 + ika \frac{Q_{m2}}{4\pi r_2}} e^{i(\omega t - k(r_2 - a) + \delta)}$$  \hspace{1cm} (3.17)

*Fig. 3-2. Geometry.*
The first of Eqs. 3.18 below represents the sound pressure at the measurement point in terms of the radial velocities of each source. The second two equations below relate the actual distances to that from the coordinate center. A helpful approximation is that the distances from the sources is large with respect to the distance between the sources, i.e., \( r \gg h \). This is properly called the geometric far field and is not related to any wavelength. The approximation is typically incorporated in the phase term but the difference in distances is often neglected to simplify calculations. The terms in red are second order approximations and are neglected. As a result, the amplitude and distance differences limit the validity of the approximation {6.1.3}.

\[
p(r, t) = \frac{ika}{1 + ika} Z_0 a e^{int} \left[ \frac{u_{m1}}{r_1} e^{-ik\theta} + \frac{u_{m2}}{r_2} e^{-ik\theta + \delta} \right]
\]

\[
r_1 = \sqrt{r^2 + h^2 - 2rh \cos \theta} \approx r + \frac{h^2}{2r} - h \cos \theta
\]

\[
r_2 = \sqrt{r^2 + h^2 + 2rh \cos \theta} \approx r + \frac{h^2}{2r} + h \cos \theta
\]

3.4.2 The Sound Intensity and Sound Power

With these approximations, the mean square pressure, intensity, and power expressions are

\[
pp^* = \frac{k^2 a^2}{1 + k^2 a^2} \left( \frac{a}{r} \right)^2 Z_0^2 u_{m1}^2 \left[ 1 + \left( \frac{u_{m2}}{u_{m1}} \right)^2 + \frac{2u_{m2}}{u_{m1}} \cos(2kh \cos \theta - \delta) \right]
\]

\[
I_r(r, t) = \frac{k^2 a^2}{1 + k^2 a^2} \left( \frac{a}{r} \right)^2 u_{m2}^2 Z_0 \left[ 1 + \left( \frac{u_{m2}}{u_{m1}} \right)^2 + \frac{2u_{m2}}{u_{m1}} \cos(2kh \cos \theta - \delta) \right]
\]

\[
W = \frac{k^2 a^2}{1 + k^2 a^2} Z_0 A_s u_{m1}^2 \left[ 1 + \left( \frac{u_{m2}}{u_{m1}} \right)^2 + \frac{2u_{m2}}{u_{m1}} \sin(2kh \cos \delta) \right]
\]

The symbol \( A_s \) represents the surface area of the sphere \( 4\pi a^2 \). If the second source is absent, the single source equation is recovered. Consider two extreme cases where the separation of the sources is small: one where they are in phase and the second where they are in opposition. The bracket terms degenerate to those shown on the right. The sound power is either quadrupled when the in-phase sources are close or cancelled when the sources are in opposite phase. The ratio of the sound power of two nearby monopoles to that of a single monopole is shown in the figures. In Figure 3-3, the sources are in phase but are of differing amplitude. In Figure 3-4, the sources are of equal amplitude, but vary in phase in 22.5 degree increments. These particular examples are for the case when \( ka \ll 1 \), so the sources act similar to point monopoles.

There are several interesting results. Consider amplitude. The minimum sound power occurs when the separation distance is \( 2kh = 11\pi / 8 \) for sources of equal amplitude, and the
reduction is 4 dB. Consider phase. Whenever the phase angle between the two sources is an odd multiple of 90 degrees, the sources add independently. It is useful to note that in (C.2) the simple diagram for the correlation of random sources shows that sources add independently whenever the correlation coefficient is zero. Phase quadrature qualifies as no correlation. The minimum output occurs at the same separation distance as for the amplitudes.

What is wrong with this analysis? The development of the equations was straightforward but the results do not coincide with reality; real sources are finite. In this mathematical exercise when the two sources merge the volumetric flow rates merge, so we must deal with their square and the sound power must quadruple. Compare this result with two finite pulsating spheres. The first flaw is that the sound from one source can pass through the body of the second. This would not be a severe handicap if the sources were very small with regard to a wavelength and the separation was large; it would be equivalent to a low frequency plane wave diffracting around a sphere. As the sources approach each other, the shadowing and scattering becomes significant. Further, it certainly is not possible to merge one physical sphere with another. Therefore the graphs above are decidedly incorrect for small values of $2kh$ and certainly for separations where $h \sim a$. The region of increased invalidity will lie to the left of the dashed lines in Figures 3-3 and 3-4. Despite this limitation, the analysis provides some insight on the interaction process.

Directivity patterns are calculated and displayed in the SoundSource program for either a single frequency or broad-band spectrum in one third octave bands. The following are the important variables:
- Separation distance
- Measurement distance
- Relative level
- Relative phase
- Source diameter
- Source velocity

The geometric far field approximation is not necessary in the software, so the measurement distance can be closer, but is restricted to distances that avoid the other restrictions noted above. The separation distance is also restricted to four times source size.
3.4.3 The Monopole above a Reflecting Plane

Often, the method above is used to determine the sound field above a reflecting plane. The method of images is used; the image source is set below the plane so that the boundary conditions on the reflecting plane are met. To do this the image source must have the same phase as the real source. For a hard surface, the amplitude of the two sources must also be equal, resulting in specular reflections (angle of incidence equals angle of reflection) and not diffuse reflections. The same cautions about the validity of the geometric far field must be applied to this model as well. The integration for sound power is done only for the real hemisphere resulting in the sound power equation for a hard reflecting surface below.

\[ W = \frac{k^2a^2}{1+k^2a^2}Z_0A_0u^2_0\left[1+\frac{\sin(2kh)}{2kh}\cos\delta\right] \]  

(3.20)

In real situations that surface may be an essentially rigid solid or an elastic fluid with a lesser impedance. If that impedance is not infinite, as assumed above, the normal velocity will not be zero. For example, a monopole may be either above or below a water surface. Not only will be impedance be finite, but the surface will support surface waves making the above equation only a first approximation. Again, the distance limitation of {3.4.3} applies here. **Key Point:** If the source is in the plane, the integration of the two sources is complete and the results of {3.10.2} apply.

3.4.4 Two Interacting Square Wave Monopoles

Fourier analysis is applied to a square wave to create a sound source with multiple frequencies. This case can be considered as a periodic volumetric flow with high output such as a loud motorcycle exhaust, so it might be approximated by a square wave. The sound pressure appears as

\[ p(r,t) = \sum_{n=1}^{L} \frac{\text{im}ka}{1+\text{im}ka}Z_0u_{m1}e^{i\omega(at-kr_1)} + \sum_{m=1}^{L} \frac{\text{im}ka}{1+\text{im}ka}Z_0u_{m2}e^{i\omega(at-kr_2)+i\delta} \]  

(3.21)

For the square wave the sums are over odd values only. Since each of the harmonics are phase matched with the fundamental, the phase term is not part of the summation. It was presumed that the surface velocity was a square wave so the harmonics decay as 1/n. Writing equations for the intensity and power is possible but not informative. Instead, the mean square pressure has been calculated in the **SoundSource** program to show how the higher harmonics create more variations in level than would be the case for a single frequency. **Key Point:** In this case there are multiple frequencies that a phase related (coherent) to the fundamental frequency and of declining amplitude. It results in more variations of output with \(2kh\) than is found in Figure 3-3. The presence of summations makes programming the equation a straightforward matter. For two interacting broadband monopoles, each of the frequencies add incoherently so the variations of output with \(2kh\) are diminished.
3.5 The Broad-band Point Monopole

It is necessary to work in the frequency domain for this case. Eq. 3.22 is an outgoing solution of the wave equation.

\[ \Phi(r, \omega) = \frac{Q_m(\omega)}{4\pi r} e^{-ikr} \]  \hspace{1cm} (3.22)

There is little new in these results when compared with the single frequency point monopole except that one works with a broad-band spectrum. The spectrum of the sound field is similar to that of the source.

3.6 The Broad-band Finite Monopole (Vibrating Sphere)

3.6.1 The Physical Variables

The radius of the sphere is \( a \). The radial velocity and volumetric flow rate spectrum at the surface is

\[ u_r(a, \omega) = U_a(\omega) \]
\[ Q_m(\omega) = 4\pi a^2 U_a(\omega) \]  \hspace{1cm} (3.23)

This leads to the following relationships

\[ \Phi(r, \omega) = U_a(\omega) \left( \frac{a}{r} \right)^2 \frac{1}{1 + ika} e^{-\hat{a}(r-a)} \]
\[ u_r(r, \omega) = U_a(\omega) \left( \frac{a}{r} \right)^2 \frac{1 + ikr}{1 + ika} e^{-\hat{a}(r-a)} \]
\[ p(r, \omega) = U_a(\omega) Z_o \left( \frac{a}{r} \right) \frac{ika}{1 + ika} e^{-\hat{a}(r-a)} \]
\[ I_r(r, \omega) = U_a(\omega) Z_o \left( \frac{a}{r} \right)^2 \frac{k^2 a^2}{1 + k^2 a^2} \]  \hspace{1cm} (3.24)

The intensity shown is the far sound field approximation. The radiation impedance remains the same as for the single frequency case. The equations are more complex spectrally, but the results are similar to the single frequency case.

3.6.2 Dimensional Analysis

By introducing dimensionless factors into the sound power equation as was done in \{3.2.8\}, the sound power can be presented in the following form.

\[ W_m(\omega) = \frac{\pi \rho_0}{c_0} \left[ \hat{Q}(\omega) S(\omega) \right]^2 U^4 L^2 \]  \hspace{1cm} (3.25)
\[ \hat{W}_m(\omega) = \pi \left[ \hat{Q}(\omega) S(\omega) \right]^2 M \]

Again, the characteristic length and speed must be interpreted as fixed values, such as mean flow speed while the dimensionless ratios represent a spectrum. It is necessary to integrate over the frequency dependent terms in the square brackets to obtain the overall sound power. There is
now not only a source strength spectrum, but also a Strouhal number spectrum. In many cases related to flow, these are dependent on Reynolds number.

### 3.7 Two Interacting Random Point Monopoles

The relationships for this case are

$$
\Phi(r, \omega) = \frac{e^{-ikr}}{4\pi r} \left[ Q_{m1}(\omega)e^{ikh\cos\theta} + Q_{m2}(\omega)e^{-ikh\cos\theta} \right]
$$

$$
p(r, \omega) = \frac{ikZ_0 e^{-ikr}}{4\pi r} \left[ Q_{m1}(\omega)e^{ikh\cos\theta} + Q_{m2}(\omega)e^{-ikh\cos\theta} \right]
$$

$$
u_r(r, \omega) = \frac{1 + ikre^{-ikr}}{4\pi r^2} \left[ Q_{m1}(\omega)e^{ikh\cos\theta} + Q_{m2}(\omega)e^{-ikh\cos\theta} \right]
$$

$$
I_r(r, \omega) = pu_r^* = \frac{k^2Z_0}{16\pi^3 r^2} \left[ 1 + \frac{i}{kr} \right]\left[ \right]
$$

$$
\left[ \right]^* = Q_{m2}^2(\omega) \left[ 1 + \frac{Q_{m2}^2(\omega)}{Q_{m1}^2(\omega)} + 2 \frac{Q_{m1}(\omega)Q_{m2}(\omega)}{Q_{m1}^2(\omega)} \cos(kh\cos\theta) \right]
$$

The geometric far field approximations of Eqs. 3.18 have been made. It is necessary to integrate the frequency spectrum to get the overall intensity. With band-limited white noise as an example {C.7} the equation is

$$
I_r(r) = \frac{Z_0}{16\pi^3 r^2 c_0^3} \left[ S_1 \int_{\omega_L}^{\omega_U} \omega^2 d\omega + S_2 \int_{\omega_L}^{\omega_U} \omega^2 d\omega + S_{12} \int_{\omega_L}^{\omega_U} \omega^2 \cos \left( \frac{\omega h \cos \theta}{c_0} \right) d\omega \right]
$$

(3.27)

The results are quite complicated and are reserved for the SoundSource program. If the second source, $S_2$, is absent we recover the usual spectrum of the monopole. If the second source, $S_2$, is equal to the first source, $S_1$, and they are incoherent ($S_{12}=0$), the third term is missing and the first two integrals are equal. Some special cases are listed below.

**Single Source Case ($S_2=0$)**

$$
I_r(r) = \frac{Z_0}{16\pi^3 r^2 c_0^3} \left[ S_1 \int_{\omega_L}^{\omega_U} \omega^2 d\omega \right]
$$

(3.28)

**Two Equal Sources, but Incoherent ($S_1=S_2, S_{12}=0$)**

$$
I_r(r) = \frac{Z_0}{16\pi^3 r^2 c_0^3} \left[ 2S_1 \int_{\omega_L}^{\omega_U} \omega^2 d\omega \right]
$$

**Two Equal, Positively Coherent Sources ($S_1=S_2, S_{12}=S_1$)**

$$
I_r(r) = \frac{Z_0}{16\pi^3 r^2 c_0^3} \left[ 2S_1 \int_{\omega_L}^{\omega_U} \omega^2 d\omega + S_1 \int_{\omega_L}^{\omega_U} \omega^2 \cos \left( \frac{\omega h \cos \theta}{c_0} \right) d\omega \right]
$$
Two Equal, Negatively Coherent Sources ($S_1 = S_2, S_{12} = -S_1$)

\[
I_r(r) = \frac{Z_0}{16\pi^2 r^2 c_0^3} \left[ 2S_1 L_1 \int_{L_1}^{L_2} \omega^2 d\omega - S_1 L_1 \int_{L_1}^{L_2} \omega^2 \cos \left( \frac{\omega h \cos \theta}{c_0} \right) d\omega \right]
\]

The broad-band point monopole case is presented in SoundSource for a source above a hard reflecting plane and in the corner of two hard reflecting surfaces. The sound pressure level as a function of angle in a quadrant is shown for a variety of physical values.

**Key Points:** The sound intensity of two interacting random monopoles is a complicated function of the spectrum of the sources. The examples above are limited examples only. The correlation between the sources largely determines the intensity.

### 3.8 Fluid Mechanical Estimates of Monopole Directivity

The big flaw in modeling the monopole is that real sources are seldom spherically symmetric, bubble, balloons, and explosions being notable exceptions. Deviations from symmetry can be estimated in two ways. In this section, the motion near the source is viewed as a streamlined fluid flow while in [3.9] known deviations from symmetry are used as models. Streamlines are normally defined for incompressible flows {1.2.3} and in many cases incompressible flow is dominant near the source. In essence, near the source the fluid mechanical Strouhal number \[ St = \left( \frac{fr}{U} \right) \] is more important than the acoustical Strouhal number.

\[
\left[ St = \frac{fr}{c_0} = St_M \right]
\]

By looking at the oscillatory streamlines near the source, it is possible to get an estimate of directivity which can be helpful in outdoor propagation. The concept is as follows. The radial component of the streamlines represents the resistive flow that results in sound. Streamlines that are primarily tangential, represent reactive flow and little sound. Streamlines that diverge more than others represent lower levels of sound in the relevant direction, while those that converge, or diverge slowly, represent higher levels of sound. Below are several examples.

#### 3.8.1 A Free Monopole

The flow in and outward from a spherical sound source is spherically symmetric as suggested in Figure 3-5. There is no preferred direction, so the sound field can be estimated to be directionally uniform. The decrease of sound pressure level with distance is suggested by the divergence of the streamlines. These obvious comments are intended as a prelude to more complicated geometries.

![Fig. 3-5. The free monopole streamlines.](image)
3.8.2 A Monopole near a Reflecting Surface

The streamlines in this example show the relationship between the near field flow and the sound field. The streamlines are shown in Figure 3-6, while the calculated sound field is shown in Figure 3-7. The calculation was for a source 2 inches in diameter, 8 inches from the surface, with at frequency of 380 Hz. The streamlines along the surface diverge more slowly than those vertical to the surface, and suggest higher levels along the surface. In the present example, there was nearly a 15 dB deviation from a perfect monopole source. An argument for the invalidity of such a simple streamline approach is the presence of lobes at higher frequencies due to wave cancellations. In many practical situations, the frequency is sufficiently low that lobes do not appear. In every case there still would be a maximum along the surface as suggested by the streamlines.

Fig. 3-6 Monopole streamlines near a hard surface.

Fig. 3-7. Directivity difference from that of a free monopole.

3.8.3 An Exhaust Pipe in Free Space

Many air handling systems exhaust through a pipe projected high above the ground. The approximate streamlines of the fluctuating mass flow from the pipe are shown in Figure 3-8. The divergence of the streamlines in the opposite direction from the flow is much greater than that along the axis strongly suggesting that the sound pressure along the axis is greater than that in the opposite direction. Measurements of the sound field surrounding a fluctuating mass flow from a pipe are shown in \{3.10.4\}. They suggest that the streamline approach has limitations (wave diffraction), especially at very low frequencies. When the duct diameter is less than about one-third a wavelength, the estimated
directivity shown in the figure is erroneous.

An alternative view of the exhaust flow is to consider the development of a ring vortex during the exhaust phase as shown in Figure 3-9. In this view the flow circulation from the vortex development induces a back flow due to the incompressible circulation around the vortex. Some of this mostly incompressible flow has a compressible component in the backward direction.

The model in Figure 3-9 is tied to that in the dipole chapter as an evolution from monopole radiation to dipole radiation \{4.x.x\}

**Key Point:** The streamline model for exhaust flow does not predict the sound field at low frequencies.

### 3.8.4 An Exhaust Pipe near a Reflecting Surface

A typical example of this geometry is the exhaust of an automobile. Figure 3-10 shows approximate streamlines. The sound in the reverse direction is expected to be much weaker, as in \{3.8.2\}. The stagnation point requires that the flow toward the surface be redirected to the right, causing the streamlines to be closer and so result in higher levels along the surface in the exhaust direction.

**Key Point:** Although no directivity data were found, it is highly likely that the sound field is a maximum on the ground along the exhaust axis.

### 3.9 Directivity of Monopole-like Sources

One of the tools in identifying the type of sound source, is its directivity. The simple streamline approach above is useful but has limitations for source identification. Below are examples of fluctuating volumetric flows from several objects showing why deviations from monopole sound field uniformity occurs. Since the volumetric flow is not radially symmetric, the sources cannot be classic monopoles; they are called *monopole-like*. Many practical problems relate to the sound emitted from flows exhausting from structures such as pipes. Since the volumetric flow rate is not radially symmetric, neither is the sound field, so the theory cannot be applied with regard to directivity. Several examples of non-uniform volumetric flows are given in this section.
3.9.1 Piston Source in a Plane Surface

A classic model in textbooks is the radiation from a circular piston embedded in a plane surface [6]. It deals with the motion of an infinite solid material at a single frequency. Mathematically, the piston is a distribution of point monopoles over the piston surface (more complex than just two monopoles {3.4}). The equations have been solved. The total volumetric flow fluctuation is the fixed velocity amplitude of the piston times its area. The back reaction of the sound from one point on other points is not considered since the motion is specified. The time delays from various points on the surface result in either reinforcement or cancellation at higher frequencies. Figure 3-11 shows the directional characteristics for three frequencies. In the far field, the level directly above the piston center remains the same (not true in the near field). The lobes at higher frequencies occur as shown. Since the figure is a two-dimensional slice of the actual pattern, the two high frequency lobes are more like donuts. At what frequency can the entire piston be treated as a point monopole with uniform directivity in the hemisphere? The directivity term for the sound pressure in the sound far field for a piston is shown in the first expression on the right. \( J_1 \) is the Bessel function of the first kind and order one (F. W. Bessel, 1785-1846). The variable \( a \) is the piston radius. Performing a series expansion of the function, and requiring that the second term be only one-tenth the first yields the inequality in the second expression. Below that value, the hemispherical sound field is symmetric; the piston acts like a single frequency finite source, despite the fact that the actual source is distributed over the entire piston face. The radiation resistance increases with the square of the frequency as with the theoretical monopole at these low frequencies. It is interesting to note that at low frequencies, the *acoustical* Strouhal number (0.14) is reasonably close to the *fluid mechanical* Strouhal number for flow over cylinders (0.20).

![Diagram](image)

\[
\frac{2J_1(ka \sin \theta)}{ka \sin \theta} \quad He = ka < 0.89
\]

\[
St = \frac{fa}{c_0} = 0.14
\]

**Fig. 3-11. Directivity of a piston source in a plane at three frequencies.**

**Key Points:** If the motion of a surface is coherent and the wavelength is long enough, the distribution of surface sources can be treated as a single source. The characteristic length is the radius of the piston and the characteristic speed is that of piston motion. Note that at low frequencies the hemispherical sound field is nearly uniform despite the fact that the volumetric flow is perpendicular to the surface. Note also that at high frequencies, lobes appear.
3.9.2 Small Piston Source in a Sphere

Another approximation to the theoretical monopole is the sound created by a piston on the surface of a sphere as opposed to that on a plane surface. It deals with the motion of a finite solid material at a single frequency. The theory has been solved [1] and some selected directivity results are shown in Figure 3-12 for a piston that is small with respect to the radius of the sphere. It is interesting to note that constructive interference can occur diametrically opposite the piston position (at 180 degrees), so that a lobe occurs behind the piston. Values of Helmholtz number \( A.2.2.2 \) near 0.5 result in nearly spherical directivity.

What happens when the piston radius is larger? The radiation impedance and the directivity pattern changes significantly [1].

**Key Points:** Fluctuating mass flows from only part of a finite structure create a non-uniform sound field. The directivity is a function of frequency so for a broad-band spectrum, the overall directivity will be determined by the spectrum contour. If the structure is finite and symmetric, it is possible for the level in the opposite direction to be higher than levels at lesser angles. This will not happen for long structures such as exhaust pipes (3.9.4). For a small piston, the characteristic length is sphere radius and the characteristic speed is that of the piston motion. For a larger piston, there is a second characteristic length, that of piston radius. Note that at low frequencies the sound field is nearly spherically uniform despite the fact that the volumetric flow is perpendicular to the sphere. Note also that at high frequencies, lobes do not appear as readily as with the piston in a plane source. Source geometry plays an important role in directivity.

---

**Fig. 3-12.** Directivity of a small piston source in a sphere at three frequencies.
3.9.3 Loudspeaker in a Cylindrical Cabinet

A loudspeaker placed at the exit and radiating from a cylindrical tube is more like that from an exhaust pipe than that from sphere, except the tube is of finite length (3.2 times the radius in the present example). Here the interest is expanded to a broad-band random noise spectrum from a solid material. There were two spectra, one had a slope of −3 dB/octave and the other had a slope of 0 dB/octave (so-called pink noise). The frequency range was from 100 to 8000 Hz; the Helmholtz number varied from \(ka=0.11\) to \(ka=9.3\). The directivity patterns are shown in Figure 3-13. When the low frequencies dominate, the maximum level difference was 4.7 dB, while for pink noise it was 11.5 dB. Because the length was finite, a small lobe was evident for the pink noise but smaller than that for the spherical source \{3.9.2\}. The characteristic length is the speaker radius and the characteristic speed is that of the volumetric flow (undetermined for this case).

**Key Points:** The directivity of a broad-band fluctuating volumetric flow depends significantly on the spectrum contour, reducing the usefulness of directivity as a tool for source identification. In many practical situations, the low frequency spectrum dominates so it is often possible to separate a monopole source from that of a dipole, the other common source type.

3.9.4 Exhaust Flow from a Pipe

Several types of sources can be present in the exhaust from a pipe. This example is restricted to a fluctuating volumetric flow at a single frequency. Levine and Schwinger provided a theoretical framework for the sound field from a pipe of semi-infinite length in a free field. Ando [17] made measurements for comparison with the theory. Although the agreement was not exact, it was close. Figure 3-14 shows the measured directivity for three values of the Helmholtz number. The characteristic length was pipe radius. The characteristic speed was that of the exhaust flow.

![Fig. 3-13. Sound pressure directivity of a cylindrical speaker.](image)

![Fig. 3-14. Directivity of cylindrical pipe sound at various frequencies.](image)
**Key Points:** The directivity of a fluctuating volumetric flow from a pipe differs significantly from that of the theoretical monopole. Sound field non-uniformity is sufficiently small to be neglected if $ka<1$ ($St<0.16$). This conclusion compares favorably with that of {3.9.1 and 3.9.2}, so that for prominent frequencies below this criterion it would be possible to use directional characteristics as a tool to identify a monopole source.

### 3.10 Modeling Monopole Sources

The basic feature of the theoretical monopole is a spherically symmetric fluctuating volumetric flow rate with a spherically symmetric sound field. Some real sources may meet these requirements such as bubble collapse, balloon pops, or the world’s largest monopole: the 50 megaton H-bomb at Novaya Zemlya. In many other cases, the fluctuating volumetric flow is highly directional, as suggested in {3.8, 3.9}, so it should be described only as *monopole-like*. Monopole and monopole-like sources can be grouped based on the physical geometry of the source:

1. **A symmetric fluctuating volumetric flow rate in free space, such as an explosion.** It creates a sound field that is symmetric and fits well with the theoretical model.
2. **A fluctuating volumetric flow rate caused by the motion of a plane surface, such as a loudspeaker.** The volumetric flow is in one direction only and symmetry of the sound field in half space can be achieved only at low frequencies.
3. **A fluctuating volumetric flow rate from a tube, such as an exhaust pipe.** The volumetric flow is in one direction only and symmetry of the sound field can be achieved only at low frequencies. If the source is a circular tube exhausting gas, the tube can have cross and spinning modes that complicate the radiation process from the opening; the exit plane no longer can be considered as a distribution of in-phase monopoles. These modes can be broad-band or a single frequency. In many of the applications modeled below, it is highly unlikely that significant cross or spinning modes exist to degrade the analysis. Spinning modes are generally associated with gas turbines.
4. **A fluctuating volumetric flow rate from more complex shapes, such as between the treads of a tire in motion.** The volumetric flow rate may be a function of angle creating a more complex sound field.
5. **A fluctuating volumetric flow rate from specifically designed surfaces, such as special loudspeakers.** To enhance the sound level in specific directions, loudspeakers are often designed with complex shapes that modify the directivity of the source.

Note that the above situations apply in a free field and are not related to contributions from reflecting surfaces.

The purpose of modeling in the next sections is not to display clean solutions to actual problems, but rather to show how to put a knowledge framework around specific sound sources by use of scaling rules and dynamic similarity principles with hope that the same technique can be used by the reader for other sound sources.
3.11 Modeling Category I Monopole Sources

Category I sources are those in which the generated sound is primarily a by-product of source motion \{1.4.1\}. Although most of the mathematical monopole models developed above are approximations to reality, it is still possible to learn much about an actual sound source using the models along with a judicious choice of the characteristic variables.

3.11.1 Combustion

It is beyond the scope of this monograph to develop the details of the sound field created by the reacting fluid motion of combustion. It is clear that the combustion of flammable gases produces a rapid expansion of the gas, but certain types of combustion do not create sound. For example, the Bunsen burner is quiet; the flame speed into the burner flow is balanced by the exiting gas speed so the flame front (density change) is stationary with regard to an observer \{2.1.1\}. The flow is laminar so there is no time component at the flame front and thus no sound. Here is a case where the density changes but no sound is created. Propane torches are similar, but have a turbulent gas flow. The turbulence creates temporal fluctuations at the flame front and thus sound. Does the collection of monopole sources distributed across the flame front create an overall monopole sound field? Measurements on a propane torch are shown in Figure 3-15. Measurements were made at two positions 90 degrees apart in the plane of the orifice, and at one position 45 degrees to the flow axis, another on the axis and one at 170 degrees from the axis. The monopole nature of the sound field is clear. Despite the fact that the flame front was somewhat shadowed by the structure of the torch, the 170 degree measurement agreed reasonably well with the others and particularly with the on-axis measurement.

**Key Points:** The symmetry of the sound field is a strong indication that all of the sources are monopole so the flame is an acoustically compact collection of monopoles distributed along the conical flame front. The source strength is the gas expansion upon combustion. The variables characteristic of the combustion process are not easily available. Here the actual characteristic lengths would be the turbulent eddy sizes proportional to the measurable burner radius. The characteristic speed would be the turbulent eddy speeds proportional to the gas efflux velocity. As a result, the characteristic variables have a spectrum of sizes and speeds resulting in a broad-band spectrum of sound.

![Fig. 3-15. Spectra of a propane torch.](image)
3.11.2 Explosions

A typical explosion will create a transient monopole field close to the theoretical, absent of any reflecting surfaces. Chemical explosions come in a variety of forms, some bounded. Weapons discharges can be considered to be transient versions of pipe exhaust flow; they can be subsonic or supersonic. Modern weapon discharges typically are supersonic, so they result in the familiar crack with little subsequent sound. Other materials can combust slowly or proceed to detonation. Typical explosive sound spectra at a distance are shown in the Figure 3-16. As a finite monopole it does not suffer from the flattening of the radiation effectiveness since the frequencies are quite low \( \text{(6.4.3)} \). The world’s biggest monopole is the hydrogen bomb. It most assuredly does not meet the linear wave equation requirements \( \text{(2.2.6, D.2)} \). The initial shock wave is succeeded by massive flows of material which create their own sound as well as creating sound by the passage of powerful winds around structures.

**Key Points:** Free explosions are inherently monopole in nature and the sound wave from them can vary from a shock to a transient sinusoid. The conversion of the stored energy to a gas defines the sound strength. Since this is a transient phenomenon, is knowing the characteristic variables useful or relevant?

3.11.3 Vergeltungswaffe Eins (V1)

This is the first example of converting the fluid mechanics of exhaust flow from a tube to a sound field. The lack of a flange makes mathematical analysis of the sound field more difficult \( \text{(3.9.4)} \). As will be seen later, there can be more source types involved in exhaust flow than the fluctuating volumetric flow rate addressed here \( \text{(6.4.4)} \). In October 1937, Werner von Braun (1912-1977) and Walter Dornberger (1895-1980) opened a weapons development area at Peenemunde, Germany. One of their weapons, created by Robert Lusser (1889-1969), was called “vengeance weapon one” in Germany, and the “buzz bomb” or “doodle bug” in England. A photograph is shown in Figure 3-17. It was flown from German controlled territory mainly to London. As long as the engine sound existed there was no problem, but when it cut out, the bomb descended. It was powered by repetitive bursts of combustion resulting in a periodic mass outflow that propelled it. The XP90 pulse jet was mounted above the aircraft body. Air entered the front of the jet tube through a louver. Fuel was added and combustion occurred, closing the intake louver and expelling the gases through the tube (about 10 inches in diameter). As pressure in the tube dropped, the louvers opened and the cycle restarted. It is clear that the mass flow into the intake and out of the exhaust would create two monopole-like sound sources, the periodic mass flow rate out of the exhaust greatly exceeding that of the inflow. There are many modern air and land vehicles that are propelled with such a device. Videos had been made of the actual
launch of the V1, probably in Belgium, and the sound spectrum at launch was recorded. A spectrum, at an arbitrary level and at an unknown angle, is shown in the Figure 3.18. Documents suggest that the pulse frequency was between 10 and 100 Hz and the data in the figure suggest it was in the 100 Hz one-third octave band. Although the harmonics show, it is clear that the flow was excessively turbulent. The engine thrust \( F \) of 600 lb, was most likely generated during flight but the number is still useful for estimates. The characteristic length is the exhaust diameter, and

\[
U = \sqrt{\frac{F}{\rho_0 \pi D^2}}
\]

\[
Q = U \pi D^2
\]

\[
W = \frac{k^2 Z_0 Q^2}{4 \pi}
\]

\[
L_p = 10 \times \log_{10} \left[ \frac{Z_0 W}{4 \pi r_p p^2} \right]
\]

Is the estimated sound pressure level accurate? No, this result applies only for the pulse frequency. Is it in the “ball park”? Yes, the remainder of the spectrum will add somewhat to the overall level. The estimate was based only on knowledge of the pulse frequency, the exhaust diameter, and the engine thrust; data which are seldom used for acoustical purposes.
Measurements of V1 sound pressure level directivity have been made and the averages are shown in Figure 3-20. The direction of flow is toward the 0 angle (exhaust) and the level on the intake axis is about 11 dB down from the exhaust axis. This figure includes all frequency components. This results shows that the intake monopole is relatively weak. The directivity results of Figure 3-20 compare well with those from Figure 3-13 for the cylindrical loudspeaker with pink noise; the higher frequencies contribute significantly to directivity.

**Key Points:** Knowing the characteristic variables and the nature of the sound source, it is possible to make a rough estimate of sound pressure levels based on some known mechanical variables.

### 3.11.4 The Vibrating Rectangular Membrane

Instead of a piston source, consider a membrane vibrating in a plane. Morse included such an analysis in his classic book [2]. The pertinent relations are

\[
\zeta_{mn} = A \sin \left( \frac{\pi m x}{a} \right) \sin \left( \frac{\pi n y}{b} \right) \cos(\omega_{mn} t) \\
\omega_{mn} = \frac{\pi}{\sqrt{\frac{T}{\sigma}}} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}
\]

The vertical displacement \( \zeta_{mn} \) is given for any position (x and y) of any (m and n) mode number in a rectangular membrane of dimensions a and b. The wave speed is determined by the membrane tension per unit length T and the mass density \( \sigma \) per unit of area of the membrane. The frequency \( \omega \) and the wave number \( k \) is determined by the wave speed and the mode shapes. For example, the \( m=n=1 \) mode is the drum mode with the center of the membrane being the maximum displacement point (upper left of Figure 3-21). The \( m=n=2 \) mode results in four areas of motion. Two of the diagonally opposite areas are in phase, while the other areas are in phase opposition (lower right of Figure 3-21).

But what about the sound radiated from this motion? To integrate the contribution from each position on the membrane is a formidable task. It is clear from the figure that each mode area causes a volumetric displacement of the surrounding medium and the phase of the next mode is opposite.

*Fig. 3-20. Directivity of V1 sound.*

*Fig. 3-21. Lower mode shapes of a vibrating membrane.*
If each mode area within a rectangular array is considered to be a monopole source the sound field velocity potential can be written as

$$\phi = \frac{Q}{\pi} e^{i\omega} \sum_{a=1}^{m} \sum_{b=1}^{n} e^{-\frac{i(kr_{ab} + \delta_{ab})}{r_{ab}}}$$  \hspace{1cm} (3.31)

The source strength can be determined by integrating the displacement over a particular mode area to get $Q = 4A$ where $A$ is the r.m.s. amplitude of each mode shape. Since the total membrane area may be large, the far geometric field assumption is not used; the distance $r_{ab}$ is that from the mode center to the measurement point. The phase term $\delta$ is either 0 or $\pi$ for alternate source positions. The equation for the mean square pressure is

$$\text{Re}[pp^*] = \frac{k^2}{16\pi^2} \frac{Z_0 Q^2}{r_{ab}^2} \sum_{a=1}^{m} \sum_{b=1}^{n} \sum_{c=1}^{m} \sum_{d=1}^{n} \cos \left[k(r_{cd} - r_{ab}) + (\delta_{cd} - \delta_{ab}) \right]$$ \hspace{1cm} (3.32)

Note that the volumetric flow rate in each mode area is assumed to be the same, although it may be different for different sets of modes. Replacing the integrals with summations must be based on the assumption that the wavelength is long with respect to the mode length; each mode area radiates as if it were a monopole. Summations lend themselves to computer modeling, and this has been done in SoundSource.

This situation is like several rectangular shaped pistons embedded in a plane. If the criterion used for the piston is applicable, the criterion for each mode to radiate as a point monopole is

$$St = \frac{fL}{c_0} = 0.28 = \frac{c_m}{c_0} \sqrt{\left(\frac{mL}{2a}\right)^2 + \left(\frac{nL}{2b}\right)^2}$$ \hspace{1cm} (3.33)

$L$ is the lesser of the two dimensions and the membrane wave speed is $c_m$. Therefore, the point monopole model would be applicable at low membrane wave speeds relative to the sound speed and at low mode numbers. The same concepts can be applied to circular membranes but Bessel functions are needed in the development.

**Key Points:** In many realistic situations, the correct but difficult integral formulation can be replaced by a summation that can be computed readily. There are two characteristic lengths, the smaller of which is the critical one. The characteristic speed is the wave speed of the membrane vibrations that define the frequency.

### 3.11.5 The Harley-Davidson Motorcycle

Owners of these motorcycles seem to love the sound created, particularly during acceleration, so the bikes often have straight pipes. If these owners bike to a rock festival, their cochlea will have a very bad day. The primary source is likely to be monopole-like above a reflecting plane due to the periodic exhaust flow \{3.4.3, 3.9.4\}. Figure 3-22 shows a passby spectrum as measured at 18 feet; the speed was close to 45 mph. The maximum of 103.2 dB occurred in the 125 Hz one-third octave band, but it is clear that the exhaust flow was very turbulent resulting in a broad-band spectrum. For comparison, Figure 3-23 shows the passby spectrum of a quieter motorcycle under the same conditions. The maximum of 75.4 dB occurred in the 125 Hz band. Note the lack of severe turbulent flow noise. Since the wavelength was much greater than the exhaust radius, the point monopole version of Eq. 3.20 was used to
calculate the approximate sound power so hemispherical uniformity is likely, except for the reflective influence of the motorcycle body. Taking into account the hard reflection, the sound power and rms volumetric flow rate was using the equations below. For the Harley the overall sound power was estimated to be 12 Watts for the Harley and 0.01 Watts for the quiet motorcycle; an impressive difference. The oscillatory volumetric flow rate that resulted in the sound was estimated for the Harley to be 3.7 ft$^3$/sec while that for the quiet vehicle was 0.22 ft$^3$/sec; a significant difference.

The contribution of tire noise (see Figure 3-24) near 1000 Hz shows in both the spectra.

These concepts can be applied to the exhaust of racing vehicles as well. The author measured the sound of Indianapolis race cars when both reciprocating and gas turbine engines were permitted. The turbines were so quiet they were booed by the race crowd, despite the fact that they were faster. They were banned. Which was the more important criterion for racing vehicles; speed or noise?

**Key Points:** To properly calculate the sound power of a vehicle exhaust, the sound pressure must be integrated over the hemisphere surrounding the source, which is only possible with the vehicle standing still and running on a drum in a proper environment. To calculate from a passby, the monopole surface reflection (3.4.3) must be taken into account, but the directivity model, which is based on the spectrum contour, is more difficult. The results in (3.9.4) may be useful. With these defined, the monopole model may be used to derive reasonable estimates of sound. Good mufflers make good neighbors (apologies to Robert Frost). When the exhaust flow fluctuation is predominantly one frequency, it is possible to estimate volumetric flow rates and exhaust velocities. The characteristic length is the exhaust diameter. The characteristic speed is that of the exhaust, but that is difficult to determine, so the vehicle speed must be used. It is likely that one is proportional to the other.

$$W = \pi r^2 \frac{P_r^2}{Z_0} 10^{\frac{L_p}{10}}$$

$$Q = \sqrt{\frac{4\pi W}{k^2 Z_0}}$$

(3.34)
3.11.6 Automobile Tire Sound

This is an example of a complex source generation situation that has had considerable study. There are a number of actual sources on an automobile, but modern design has reduced all but the noise from tires (except for certain vehicles with predominant exhaust noise). Years ago there was a question about the nature of the sound from tires. Was it casing or tread noise? Figure 3-24 shows a typical one-third octave band sound spectrum of a “quiet” automobile moving at about 45 mph. The frequency at the maximum is in the 1000 Hz one-third octave band. The mass continuity equation suggests that the Strouhal number be of order one, so using Strouhal modeling, with 0.1<St<1, the characteristic length L is found to be between one-sixteenth inch and three quarters of an inch. This result strongly suggests that the source is not the casing. The low frequency maximum in the spectrum, near 63 Hz, is likely related to engine RPM and therefore to exhaust noise \(3.11.5\). From this modeling, it seems clear that the characteristic length is tread size and the characteristic speed is vehicle speed. The spectrum for a large pickup truck has a maximum in the 800 Hz band at similar speeds, reinforcing the tread as the characteristic length since it has larger and deeper treads.

What happens to the treads? Recent research [18] has suggested that there are four mechanisms for tire noise: air pumping, tread block motion, tread contact, and tread separation.

Air pumping is compression of the air between treads; the treads must be compressed and distorted when encountering the road surface. Some of that action results in compression of the air between the treads which should result in monopole-like sources. The level of the sources should be a maximum lateral to the vehicle direction. Tread block motion is the deformation of the tread bodies themselves while in contact with the pavement. This is likely a dipole source if the correlation between them is substantial \(4.9.6\). Tread contact and separation occurs as the treads first contact the road and then leave it. These events must be monopole-like, but out-of-phase, so may be like two nearby monopoles that act like dipoles at low frequency. Their levels should be a maximum in the direction of vehicle motion. The shearing of the tread block should result in quadrupole sources, but the relative inefficiency of these latter sources \(5.1\) suggests that their contribution is small. Road roughness adds to the source strength for each of the above mechanisms. Figure 3-25 [19] shows the speed dependence of tire noise.
noise from both trucks and automobiles. The data compare very favorably with the $U^4$ law of \{3.2.8\}: a 12 dB increase per doubling of speed.

Figure 3-26 [18] shows a more comprehensive set of data for the speed dependence of automobiles and trucks and for a variety of road conditions and measurement methods. The heavy line is the monopole speed dependence. The scatter in the data makes the mean lines unreliable for exact comparison, but the trends are clearly more like a monopole source than that of a dipole (18 dB per doubling of speed). Trucks with larger treads create more “pumping” sound (volumetric flow rate fluctuations) and fit the monopole model reasonably closely. A local test was performed. One automobile was driven by at two speeds; 30 and 60 mph over the same road. The maximum passby spectra were measured. The dynamic similarity equation (Eq. 3.10) was used to adjust the high speed data in both level (U correction) and frequency (Strouhal correction). The two spectra are shown in Figure 3-27. The close match of the adjusted high speed data to the low speed data at the maximum strongly supports the monopole nature of the source and thus tread air pumping as the predominant sound source. The spectrum slope above the maximum for these two samples do not agree closely, suggesting that the source strength term in Eq. 3.10 may be speed (Reynolds number) dependent at higher frequencies. It is interesting to note that the spectrum slope above the maximum level (for much of the data taken) very closely followed the -9 dB/octave slope ($f^3$ dependence) as was found for several other sources.

**Key Points:**

Strouhal modeling was used to define the characteristic length (source size). The speed dependence was used to define the source type and the characteristic speed; vehicle speed. To reduce tire noise based on the monopole model, the motion (distortion, compression) of the treads must be such as to minimize the mean square of the rate of volumetric flow, or mass flow, rate between the treads.

\[
I_m(r) = \frac{Z_0}{16\pi^2 r^2 c_0} \left( \frac{\partial Q_{\infty}}{\partial t} \right)^2 = \frac{1}{16\pi^2 r^2 Z_0} \left( \frac{\partial M}{\partial t} \right)^2
\]
There are two monopole configurations. The first is air pumping perpendicular to the direction of travel; it is likely to have a maximum level in that direction. The second is the contact/separation event; it is likely composed of two monopole-like sources out-of phase and with a maximum level likely to be in the direction of travel. Separation of these types might be done by measurement of treadless tires such as those used in race cars.

### 3.11.7 Bubble Formation/Collapse

The formation or collapse of a bubble should represent a transient monopole in a relatively free field. High speed data on bubbles in water suggest very high collapse rates and large expansion wave pressures, so bubble collapse can create significant sound. Submariners have always been concerned about propeller cavitation noise so there has been considerable study of bubble collapse. If bubble size is used as the characteristic length, and the collapse rate as the characteristic speed, it would suggest that bubble frequencies are predominantly at high frequencies. However, bubble size is not always constrained to be small. Any form of cavitation or boiling will result in a broad-band spectrum. Boiling water in open pots, for example, has been known to create almost pink noise. The spectrum is the result of a distribution of bubble sizes and formation/collapse rates. It is likely that in a free field the overall directivity of all bubble motion will be close to the classic monopole, similar to 3.11.1

**Key Points:** Bubbles are inherently monopole sound sources. The characteristic length is maximum bubble diameter and the characteristic speed is the rate of formation or collapse. These characteristic variables should be used in any research; they will have a distribution that defines a sound spectrum.

### 3.11.8 Splashes

The sound of liquid droplets falling on a surface depends importantly on the character of the surface. If the surface is a large, stiff and massive (highly inelastic) solid, any sound created is due solely to fluid motion. If the surface is an elastic solid, such as a thin metal plate, the spectrum of the sound depends critically on the mode of vibration excited by the drop. If the surface is a fluid, not only are surface waves created but the possibility of bubble formation\ collapse occurs. As with bubbles, the characteristic length is droplet diameter and the characteristic speed is that of each droplet. For one drop, the spectrum is well defined and the radiation should be monopole-like {3.9.1}. Multiple droplets are addressed in 3.11.9.

One technique to reduce splash noise is the introduction of a screen to break droplets into very small droplets. The technique reduces the characteristic length (droplet size) and the characteristic speed (droplet speed) and thus shifting both frequency and sound level. It is used to quiet home washing machines, and was the subject of a 19th century US patent where the lifting of a toilet cover deployed a screen across the toilet.

**Key Points:** The sound from liquid splashes on liquid surfaces can be interpreted as a monopole-like sources located at the surface. The characteristic length is droplet size and the characteristic speed is droplet speed. Multiple droplets create a spectrum of characteristic variables.
3.11.9 Waterfall Noise

Waterfalls are examples of complex sound sources based on splashes. Typically, the sound results from both bubble formation\textsuperscript{3.11.7} and droplet impact \textsuperscript{3.11.8}. A waterfall is a massive collection of droplets falling on an already disturbed water surface. Consider the impact process. Some waterfalls start with laminar flow, others are already turbulent. The characteristic length (droplet size) and characteristic speed upon impact will depend on the nature of the initial flow; the height of fall and the point at which flow transitions to turbulence. Figure 3-28 shows the sound spectrum of a fifty foot wide waterfall with a fourteen foot drop measured at a distance of fifty feet. The starting flow was turbulent. Significant sound occurs at very high frequencies and the spectrum is very similar to white noise (the low frequency deviation of the data is associated with local road traffic noise). Free fall (no drag) estimates give a maximum impact speed near 30 ft/sec for a droplet. It is likely that air drag of droplets is strongly diminished due to air entrainment by the large number of droplets, so that number might not be too high as a characteristic speed. Droplets were on the order of 0.1 inches. Using a Strouhal number range from 0.1 to 1, the spectrum maximum should be between 360 and 3600 Hz. Obviously, there is more to waterfall noise than simple impact. Bubble formation and collapse is the other plausible possibility \textsuperscript{3.11.7}. Further, waterfalls have linear extent, so the sound field will depend on the lineal correlation of the various droplets. Falls that remain laminar for a large percentage of the fall will likely decay with distance more like a line source, while larger, turbulent falls will decay more as a linear collection of independent sources.

Lilly \textsuperscript{20} has measured the sound from a waterfall in Bellevue, Washington Park, 200 feet wide and with a 9 foot drop and apparently starting with laminar flow (Figure 3-29). At a measurement distance of 12 feet, the spectrum had a 1 dB/octave slope up to 8000 Hz (nearly pink). This result suggests the characteristic length was larger, and the lack of very high

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{waterfall_spectrum.png}
\caption{Sound spectrum of a waterfall.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{various_waterfall_spectra.png}
\caption{Various waterfall spectra.}
\end{figure}
frequencies suggests that the impact process of the lower fall distance reduced the severity of bubble formation. A number of sound spectra of Niagara Falls at similar distances were taken from a video and averaged and added to Figure 3-29. The levels were arbitrarily adjusted to fit so the spectral contours could be compared. At the present time there are insufficient data to scale waterfall spectra.

**Key Points:** Although it is clear that droplet size is the characteristic length and droplet speed is the characteristic speed, the spectra of neither are readily available. The interaction of the splash adds additional complexity to waterfall noise.

### 3.11.10 Electrical Discharges

All electrical discharges result in rapid temperature and density changes that propagate as sound. Sparks are monopole sources while corona discharges may be line sources. Electrical discharges are not covered in this monograph.

#### 3.12 Modeling Category II Monopole Sources

Category II sources are those in which the generated sound has a strong impact on the source that creates the sound \{1.5.2\}. Again, it is still possible to learn much about a sound source using the monopole models along with the scaling rules and judicious choice of the characteristic variables.

#### 3.12.1 The Hole Tone (Tea Pot Whistle)

The steady flow from a circular orifice can be converted to an oscillatory flow by adding a downstream plate with a circular hole aligned with the orifice. Small perturbations in the flow are fed back to the orifice to cause a variable volumetric flow rate through the hole because of the symmetry of the feedback. The disturbance is amplified along the jet \{1.5\} forming the feedback loop when the disturbance encounters the hole. Figure 3-30 shows a schematic of the geometry. The oscillatory volumetric flow through the plate hole creates a nearly pure tone monopole-like sound field. This phenomenon is one of many that have been called aerodynamic whistles. The frequency depends on the steady flow rate from the primary orifice and is subject to jumps as the phase relationship of the feedback changes. One part of the feedback loop is the incompressible disturbance wave carried with the steady flow from orifice to plate. Since the gap between plate and orifice is typically a small fraction of an acoustical wave length, the other part of the feedback loop is the hydrodynamic field of the monopole source. In a study [21], the characteristic speed was chosen to be the average speed of the jet at the orifice (deduced from measured volumetric flow rate). The characteristic length was chosen to be the orifice diameter. Figure 3-31 shows the Strouhal

![Fig. 3-30. Hole tone geometry.](image-url)
number as a function of the Reynolds number based in these dimensions. The hole and orifice diameter \( \delta \) for this figure was 5 mm, and \( h \) was the separation distance between plates, in mm. The use of Strouhal number showed most clearly the almost linear relation between frequency and speed. The nonlinearity of the feedback system is exemplified by the hysteresis loops for each gap ratio.

More would have been learned had the characteristic variables been chosen so the hysteresis loops overlapped. The frequency is determined by the plate separation \( h \) and the wave speed in the jet, which is a fraction of the exit speed. Similarly, the source strength is determined by the oscillatory flow rate at the plate hole. The area of the hole is equal to the orifice diameter \( \delta \) and the flow speed through the hole is a fraction of the orifice speed. There are now two characteristic lengths, \( \delta \) and \( h \), and two characteristic speeds, \( \alpha U \) and \( \beta U \). The variable \( \alpha \) is the ratio of the jet wave speed to the orifice speed; it must be tied to the spreading of the jet and so must be a function of \( h \), the separation distance. The variable \( \beta \) is the ratio of the mean speed at the hole to the orifice speed also a function of \( h \). These characteristic variables can be entered into the dimensionless sound power equation of Eq. 3.10 to yield Eqs. 3.35. By adjusting the vertical height of the data, a reasonable value of \( \alpha \) can be determined. By adjusting the horizontal value of the data, a reasonable value of \( \beta \) can be determined. The best fit is when the hysteresis loops nearly coincide. When this is done there is a considerably better overlap of the data, from which any deviations from dynamic similarity can be determined.

The uniformity of the measured sound field for this source confirmed the monopole-like nature of the source; more correctly it is more like a piston source in a plane \( \{3.9.1\} \). Measurements of speed dependence showed it to be very close to \( U^4 \), further confirming the monopole nature of the source. Because feedback controlled the sound output, the measurements of speed dependence could only be made in the regions of stability between jumps. At higher Reynolds numbers, the flow became chaotic resulting in broad-band sound. Rayleigh (J.W. Strutt, 1842-1919) was aware of the hole tone phenomenon; it was called the "bird call" then.

**Key Points:** This is a case of examining the physics of the situation to arrive at lengths and speeds that are more characteristic of the sound generation process than average values, such as jet exit speed. By adjustment of measurement results, otherwise arbitrary ratios can be determined to provide more insight into the phenomenon. Both the directivity and speed dependence indicate the monopole nature of the source and so permit a dynamic similarity.
equation (Eq. 3-35) to be developed. Although this phenomenon occurs at low Reynolds numbers, there seems to be some evidence that a similar event occurs on aircraft landing gear covers with holes and spinning disks with holes.

3.12.2 Hartmann Whistle (Stem Jet)

The previous example was relevant to low speeds, this example pertains to supersonic speeds. When a subsonic jet impinges on a cavity, jet instability becomes part of the feedback loop. When a supersonic jet impinges on a cavity, shock wave instability becomes part of the feedback loop. The Hartmann whistle (Figure 3-32) is one example of the latter case. A cylindrical cavity with one end open and facing the supersonic circular jet will result in extremely intense sound. The shapes in the figure represent the shock/expansion cells within the jet. A related configuration, called the stem jet, has a central rod in the jet runs through and that supports the cavity. There are a number of other geometric variations, all of which operate in similar fashion. These devices have been studied [22] and reviewed Raman [23]. Here we look primarily at the Hartmann whistle.

The shock cells of the jet interact with the shock in front of the cavity (the flow in the cavity being subsonic). Small symmetric disturbances in the jet stream are amplified as they proceed toward the cavity (similar in some respects to the hole tone) causing the shock in front of the cavity to oscillate. The shock front acts much like a piston source of high energy resulting in a monopole-like sound field. Again the volumetric flow is unidirectional unlike the theoretical monopole. The sound field may be similar to that created by oscillatory flow from a pipe, except for presence of the supersonic jet structure which can strongly modify the directivity. The original equation of Hartmann (1881-1951) [24] is the first of Eqs. 3.36.

\[
\frac{\lambda}{d} = 5.8 + 2.5 \left\{ \frac{h}{d} - \left( 1 + 0.041(P - 0.9)^2 \right) \right\}
\]

\[
St = \frac{fd}{c_0} = 0.17 \approx \frac{fh}{U}
\]

(3.36)

The diameter of the orifice and cavity being \(d\), the distance between orifice and cavity being \(h\), and the orifice pressure \(P\) was given in kg/m². At the lower limit of \(h\) the second term disappears. In this case, the equation could have been reformatted in terms of a Strouhal number as shown in the second equation of Eqs. 3.36. The characteristic speed is shown as that of sound (which is also \(U\) at the nozzle exit). It is interesting that the number is very close to that found by Strouhal for flow over a cylinder \{4.10.1.1\}. There are two characteristic length scales. Nozzle diameter \(d\) characterizes the sound power while the separation distance \(h\) characterizes the frequency.
Comprehensive studies of this phenomenon [24-26] have shown that the position of the cavity is critical in creating sound. The process has hysteresis loops and the frequencies are related to multiples of the quarter wavelength resonance of the cavity. Reference 23 has a comprehensive discussion of the phenomenon and is recommended for those interested in the physics of the process and the applications of the whistle.

After reformatting Hartmann’s formula (the first of Eqs. 3.36), and using Eq. 3.10, we get Eqs. 3.37 for sound power. Since the characteristic speed $U$ and $c_0$ are essentially the same, it can be expressed as the second equation. Although the amplitude factor $A$ replaces the dimensionless volumetric flow rate in these equations, the speed dependence strongly confirms the monopole-like characteristics of the Hartmann whistle.

$$W_h = \frac{\rho \pi d^2}{2} (2\pi f a)^2 c_0 = A \rho f^2 d^2 h^2 c_0 = A \frac{\rho}{c_0} \left( \frac{f d}{c_0} \right)^2 c_0^4 a^2$$

$$W_h = A \frac{\rho}{c_0} S \pi U^4 L^2$$

A cousin to the Hartmann whistle is shown in Figure 3-33; it is called the Galton whistle. Here the cavity is excited by an annular jet which oscillates symmetrically around the sharp edges of the cavity. It appears to be a circular version of the edge tone (4.10.5) which is a dipole source. Since it is highly likely that the oscillations are coherent around the periphery, there should be a fluctuating volumetric flow rate from the cavity with only a small net lateral force. Thus the source is yet another version of a monopole-like geometry; the volumetric flow rate is a cylindrical area between the jet and cavity.

**Key Points:** As with the earlier examples, it is possible to determine the characteristic variables to provide valuable insight about a sound source. For complex phenomenon such as these whistles, such insight is insufficient to lay bare the detailed workings of the mechanism and leads only to suggestions on which variables are important for future research.

*Fig. 3-33. Galton Whistle.*
3.12.3 Corrugated Pipe Tone

Pipes with sinusoidal variations of radius are used to permit bending. Steady flow through the pipe at low Reynolds Numbers results in a fluctuating volumetric flow rate that generates a monopole-like sound field at the pipe exit \([3.9.4]\). Examples of such pipes are shown in Figure 3-35. The plastic pipe is actually a child’s toy that sounds when the pipe is whirled around. The metal pipe shown was actually used in the Concorde to provide cooling air to the pilots. It was supplied to the author by John Ffowcs-Williams. The tone was unacceptably loud to the pilots, so it was replaced with a straight cylinder.

It is similar to the hole tone in that it is due to a periodic volumetric flow rate; is subject to frequency jumps and hysteresis loops but the feedback is internal. The characteristic speed is the mean flow through the pipe and the characteristic length must be a multiple of the spacing between corrugations. Although there is no evidence that this device has been studied in detail, it is likely that the flow instability, being weaker at low speeds, needs to travel several corrugations to establish the feedback loop. As the speed increases, the loop can be established with fewer corrugations.

Simple tests were performed on the yellow plastic tube. The corrugations were 0.011 feet apart. Without knowledge of the Strouhal number or flow rate, the similarity relationship \(StU = f_r n L\) was used to determine the number of relevant corrugations. \(L\) is the corrugation spacing and \(n\) is an integer. The highest frequency (7554 Hz) was found in the “overblown” condition and \(n\) was presumed as one corrugation. At the least flow rate, the frequency of 2452 Hz compared favorably to \(n=3\). At intermediate flow rates, several non-harmonically related frequencies occurred simultaneously suggesting that several corrugations were involved in the sound generation. In the smaller metal tube, a predominant tone appeared at 6174 Hz and corresponded to \(n=2\).

**Key Points:** The characteristic length for this device is now \(nL\), where \(n\) is close to an integer and \(L\) is corrugation spacing. Similar to the hole tone analysis, the characteristic speed is that of the disturbance which is likely a fraction of the mean speed through the tube. A unique aspect of this sound source is that the flow carries both the wave amplification and the feedback signal internally. More needs to learned about this device.

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![Flow in corrugated pipes can create single frequency tones.](image-url)
3.12.4 Pipe Tone (Pfeifenton)

A cylindrical cavity with a small circular, square-edged hole at one end and totally open at the other is known to generate a tone when air is passed through it. It is subject to frequency jumps and hysteresis loops similar to the hole tone \[3.12.1\]. The unique feature of this device is that the tone sounds only with flow into the cavity; it is an acoustical diode. It can be blown from the orifice side or suction can be applied from the open end. The fundamental tone occurs near \(\lambda=4L\), so one characteristic length \(L\) is that of the tube. The characteristic speed \(U\) is that of the flow through the hole. A monopole-like sound field is generated by motion at either the small or larger end, depending on which end is open. Karthik [27] and Anderson [28] have studied this phenomenon and concluded that symmetric vortex shedding on the cavity side is the driving agency. An example of this device is shown in Figure 3-35; it had a 1/8 in. diameter hole, was 1.9 inches long, and 0.8 inches in diameter. The quarter wave resonance was calculated to be 1780 Hz, while the measured fundamental was 1625 Hz with detectable second and third harmonics. End corrections for radiation from the openings is needed to bring the two frequencies in consonance. A second and third characteristic length is the diameter of the orifice \(d_1\) and the diameter of the tube \(d_2\). They determine the end corrections.

**Key Points:** In most situations where a resonant cavity is in conjunction with a flow field, it does not contain mean flow \[3.12.2\]. Here, as with the corrugated pipe tone, the cavity contains both flow and sound. The most unique aspect of this device is that the tone sounds only with flow in one direction.

### 3.12.5 Thermal Sources

As noted in Chapter 2, fluctuating temperatures can be a source of sound. There are a number of Category II sources that occur in practical geometries. Most were discovered in the 19th century and are generally associated with resonant structures, such as tubes. The density change caused by heat release can be significant, so the generated sound field also can be significant.

Pieter Rijke (1812-1899), a Dutch physicist, heated a gauze material inside a vertical tube. The heat transferred to the air in the tube set it into near half-wave resonance. It is now called the Rijke tube. For a tube about one meter long and 3.5 cm in diameter, it was estimated that the power was equivalent to a loudspeaker with 1 kW of input at about 170 Hz. Originally, the gauze was heated with a Bunsen burner. Later, a wire grid was heated electrically. Heating cool air as it passes over the hot source sets the resonance when the phase (gauze/grid location) is correct. Warm air passing over cold wire grids will accomplishes the same objective.

What are the characteristic lengths and speeds? The characteristic length is heat source position \(aL\), where \(L\) is the tube length. The characteristic speed must be that of the air flowing
over the heat source. That speed is the acoustic motion plus or minus the heated/cooled convection speed. References to this source can be found [4]. Since the resonance is about half-wave, the sound field is from two in-phase monopole-like sources, one at either tube end.

A gas flame inside a tube can drive resonance. It was also described in the 19th century and is called a *singing flame*. The small fluctuations in gas flow from the burner are amplified by the resonance and with appropriate phase results in significant energy release and monopole-like sound from the tube.

**Key Points:** Heat release can be a potent source of sound. The few examples given above amplify the potential importance of heat transfer in creating sound sources.
Chapter 4
Dipole Sources

4.1 The Mathematical Model

The wave equation for the dipole includes dependence on one angle.

\[
\frac{\partial^2 \psi}{\partial \psi} = 0
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} (r \phi) = 0 \tag{4.1}
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + k^2 \Phi = 0
\]

\(\phi(r, \theta, t)\) and \(\Phi(r, \theta, \omega)\) are the velocity potentials. The polar angle \(\theta\) needs to be taken into account for this case. A harmonic solution is

\[
\phi(r, \theta, t) = \frac{Q_d}{4\pi} \frac{1 + ikr}{r^2} \cos \theta e^{i(\omega t - kr)} \tag{4.2}
\]

The key physical variables are

\[
p(r, \theta, t) = \rho \frac{\partial \phi}{\partial t}, u_r(r, \theta, t) = -\frac{\partial \phi}{\partial r}, u_\theta(r, \theta, t) = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}, u_r(r, \theta, t) = 0 \tag{4.3}
\]

4.1.1 The Dipole as a Merger of Two Monopoles

Another way to view the solution is to differentiate the monopole velocity potential in a given direction, say \(z\), where \(\theta\) is the angle from the \(z\) direction.

\[
\phi_d(r, t) = \frac{\partial}{\partial r} \left( \frac{Q_m}{4\pi r} e^{i(\omega t - kr)} \right) \cdot \frac{\partial r}{\partial z} = \frac{Q_d}{4\pi} \frac{1 + ikr}{r^2} \cos \theta e^{i(\omega t - kr)}
\]

\[
2r \frac{\partial r}{\partial z} = 2z
\]

\[
\frac{\partial r}{\partial z} = \frac{z}{r} = \cos \theta \Rightarrow \frac{\partial r}{\partial \theta} = \frac{r}{r} = \cos \theta \tag{4.4}
\]

4.1.2 Interpretation of the Source Strength

The quantity \(Q_d\) is called the dipole source strength and the dimensions are \([L^4/T]\); not very useful for physical intuition so it must be interpreted. Although the dimensions of \(Q_d\) are not those of volumetric flow rate, the question remains whether there is a component of
oscillatory outflow associated with it. Performing an integration around the source in the near sound field yields

\[ Q^2 = \int_0^{2\pi} \int_0^\pi u(r, \theta) \sin \theta d\theta d\phi = \frac{Q}{r} e^{i\omega t} \int_0^{2\pi} \cos \theta \sin \theta d\theta = 0 \]  
(4.5)

*The net volumetric flow rate is zero.* Next, look at the force acting on the medium. The component of force on a spherical surface in the \(z\) direction (\(\theta = 0\) direction) is determined by integration of the \(z\) component of the pressure in the near acoustic field.

\[ F_z = \int_0^{2\pi} \int_0^\pi p(r, \theta) \cos \theta \sin \theta d\theta d\phi = \frac{ikZ_0Q_d}{3} e^{i(\omega - kr)} \]  
(4.6)

\[ Q_d = -\frac{i3F_z}{kZ_0} \]

It is non-zero and is related to the dipole source strength \(Q_d\). The component of force in the plane *normal* to the \(z\) direction is

\[ F = \int_0^{2\pi} \int_0^\pi p(r, \theta) \sin \theta \sin \theta d\theta d\phi = 0 \]

Using the same method as for the monopole \(3.1.1\), the sound power of a dipole can be expressed simply

\[ W_d = \frac{3}{4\pi Z_0 c_0^2} \left( \frac{\partial F}{\partial t} \right)^2 \]  
(4.7)

*The sound from a dipole is created by the mean square of the time rate of change of force acting along the dipole axis.*

### 4.2 The Single Frequency Point Dipole

#### 4.2.1 Physical Variables

The variables of the point dipole, in terms of force, are shown in Eq. 4.8 below.

\[ \phi(r, \theta, t) = \frac{3F_z}{4\pi kZ_0} \frac{1 + ikr}{r^2} \cos \theta e^{i(\omega - kr)} \]

\[ u(r, \theta, t) = \frac{3F_z}{4\pi kZ_0} \frac{1 + ikr}{r^3} \sin \theta e^{i(\omega - kr)} \]

\[ p(r, \theta, t) = \frac{i3F_z}{4\pi} \frac{1 + ikr}{r^2} \cos \theta e^{i(\omega - kr)} \]

\[ s = \frac{i3F_z}{4\pi c_0 Z_0} \frac{1 + ikr}{r^2} \cos \theta e^{i(\omega - kr)} \]  
(4.8)

\[ u_r(r, \theta, t) = \frac{3F_z}{4\pi kZ_0} \frac{2(1 + ikr) - k^2 r^2}{r^3} \cos \theta e^{i(\omega - kr)} \]

\[ T = \frac{1}{R \gamma} \frac{(\gamma - 1)}{4\pi kZ_0} \frac{1 + ikr}{r^2} \cos \theta e^{i(\omega - kr)} \]

All the variables have two components; one at quadrature with the other. The presence of the cosine and sine functions implies phase reversals in the different hemispheres. The pressure
and radial velocity form of two ovoid spheres located on each side of the z axis; one is out of phase with the other. The lateral velocity is phase shifted from the radial velocity and is in the form of a doughnut in the x-y plane.

4.2.2 Near Field

For a fixed force and in the very near field, the pressure and velocity increases without limit as the source is approached. This suggests the limitation of “point” source analysis and possibly the invalidity of the wave equation on which the analysis is based \(\{6.1.1\}\). Since both the radial and lateral velocities are at quadrature with the pressure, the motion is mainly reactive, the flow being primarily hydrodynamic.

4.2.3 Far Field

For a fixed force and in the far field, the radial velocity is in phase with the pressure (as in the plane wave case). The radial intensity is now completely resistive. Note that the sound pressure increases indefinitely with increase in frequency.

4.2.4 Radiation Impedance

The radial impedance is a more complicated function of distance than for the monopole. In the far field, the impedance approximates that for the plane wave. In the near field the impedance diminishes with decreasing distance; it is difficult to create a simple physical explanation for a point source.

\[
Z_r = \frac{p}{u_r} = Z_0 \frac{kr(k^3 r^3 + i(2 + k^2 r^2))}{4 + k^4 r^4}
\]

\[
Z_0 = \frac{p}{u_\theta} = Z_0 (ikr \cot \theta)
\]

(4.9)

4.2.5 Sound Intensity

The radial sound intensity is a more complicated function of distance than for the monopole. The intensity is shown on the left hand in Eqs. 4.10. The far field approximation is shown on the right and it decreases with the square of the distance. Note that it continually increases with the square of the frequency; difficult to justify physically. In the near field, it increases without limit as the source is approached.

\[
I_r (r, \theta) = \frac{9F_z^2}{16\pi^2 kZ_0} \left[ \frac{k^3 r^3 + i2(1 + k^2 r^2)}{r^5} \right] \cos^2 \theta
\]

\[
I_r (r, \theta) = \frac{9k^2 F_z^2}{16\pi^2 r^2 Z_0} \cos^2 \theta
\]

(4.10)
4.2.6 Estimates of Sound Intensity

Direct intensity measurements and their integral to calculate sound power require specialized equipment. How well does a sound pressure measurement approximate intensity for a point dipole? The relationship for this is

\[ \text{EST}[I_r] = \frac{p}{Z_0} \frac{p^*}{Z_0} = \frac{9F_z^2}{16\pi^2 Z_0 r} 1 + k^2 r^2 \cos^2 \theta \]

A mean square pressure measurement will not represent the intensity correctly at all distances. The criterion for a valid estimate is that \( kr >> 1 \). The specific value for a 1 dB error is given in \( \{6.1.2\} \). On the source axis in the far field, the sound pressure level and source strength relationships will be

\[ L_p = 10 \log_{10} \left[ \frac{9k^2 F_z^2}{16\pi^2 r^2 p_R} \right] \]

\[ F_c = \frac{2rp_R c_0}{3f_0} 10^{\frac{L_p}{10}} \]

The following relationships result:
- 6 dB/octave increase per doubling of frequency
- 6 dB increase per doubling of source strength
- 6 dB decrease per doubling of distance

Note that the above results imply that the sound pressure level will increase indefinitely with frequency - pure nonsense. At some point, the wavelength will be on the order of the actual source size. This is taken into account in \( \{4.3\} \) below. In many practical cases, this is not an issue, but it is important to be aware of this limitation \( \{6.1.4\} \).

4.2.7 Sound Power

The sound power is the integral of the intensity in the far sound field as shown below.

\[ W_d = \frac{3k^2 F_z^2}{4\pi Z_0} \]

(4.11)

4.2.8 Dimensional Analysis

Adding the dimensionless force \( \{A.2.2.7\} \) and dimensionless frequency (Strouhal number \( \{A.2.2.1\} \)) to the point source sound power equation results in Eqs. 4.12. If dynamic similarity is achieved, these two dimensionless factors remain constant resulting in a \( U^6 \) speed dependence and a dependence on size squared \( L^2 \). These are the two characteristic variables that need to be defined for any source.

\[ W_d = \frac{3k^2 F_z^2}{4\pi Z_0} = \frac{3\pi p_c}{c_0^3} F^2 S^2 U^6 L^2 \]

(4.12)

\[ \hat{W}_d = 3\pi F^2 S^2 M^3 \]

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The Strouhal number remains relatively constant for many physical systems. The dimensionless force need not be constant however. The characteristic speed may be a fluctuating speed proportional to a steady speed, such as in fluid sound sources. The ratio may vary over a broad range of speeds. It is often a weak function of the Reynolds number, so the two dimensionless numbers may be considered constant, but only as a first approximation. Deviations from the speed law are discussed in \{6.3\}. The value of these deceptively simple equations is that the nature of the source can be determined by varying the speed, or alternatively, knowing the source type, predict the speed dependence. Predictions based on this equation are 6 dB per doubling of size and 18 dB per doubling of speed. These equations are used in the examples of \{4.9\} and \{4.10\}.

Eqs. 4.12 apply strictly to a single frequency point source. However, the concept can be applied to a broadband frequency spectrum \{4.5\}. Since all sources are of finite size; how large must a source be to invalidate these equations? That is addressed next.

**Key Points:** Since the speed dependence of a dipole source is very high, it is critical to identify such source.

### 4.3 The Single Frequency Finite Dipole (Oscillating Sphere)

#### 4.3.1 Physical Variables

All real sources have finite dimensions. Having the source finite removes the singularities (and possibly the approximation errors) of a point source. The finite dipole is a rigid oscillating sphere of radius $a$ moving along the dipole axis ($z$-direction). The boundary conditions on the surface of the sphere will determine the resulting sound field. Since we are concerned with a total applied force, $F_a$, to the sphere, it is necessary to integrate the component of pressure in the $z$ axis direction to get that force. To do this the pressure relationship for the point source is used, but with an unknown constant $A$. To satisfy this equation:

$$A = \frac{3F_e e^{ika}}{4\pi (1 + ika)}$$

The phase reference is the radius of the sphere and the important quantities are

$$\phi(r,t) = \frac{3F_z}{4\pi r^2 kZ_0} \left( \frac{1 + ikr}{1 + ika} \right) \cos \theta e^{i(\omega t - k(r-a))}$$

$$p(r,\theta,t) = \frac{i3F_z}{4\pi r^2} \left( \frac{1 + ikr}{1 + ika} \right) \cos \theta e^{i(\omega t - k(r-a))}$$

$$u_r(r,\theta,t) = \frac{3F_z}{4\pi kr^2 Z_0} \left[ \frac{2(1 + ikr) - k^2 r^2}{(1 + ika)} \right] \cos \theta e^{i(\omega t - k(r-a))}$$

$$u_0(r,\theta,t) = \frac{-i3F_z}{4\pi kr^3 Z_0} \left( \frac{1 + ikr}{1 + ika} \right) \sin \theta e^{i(\omega t - k(r-a))}$$

$$s = \frac{i3F_z}{4\pi c_o r^2 Z_0} \frac{(1 + ikr)}{(1 + ika)} \cos \theta e^{i(\omega t - k(r-a))}$$

$$u_r(r,\theta,t) = \frac{3F_z}{4\pi kr^2 Z_0} \left[ \frac{2(1 + ikr) - k^2 r^2}{(1 + ika)} \right] \cos \theta e^{i(\omega t - k(r-a))}$$

$$T = \frac{1}{R \gamma} \frac{i3c_o F_z}{4\pi kr^2 Z_0} \left( \frac{1 + ikr}{1 + ika} \right) \cos \theta e^{i(\omega t - k(r-a))}$$

#### 4.3.2 Near Field

In the near sound field, the pressure decreases with the square of the distance from the source, while the radial velocity decreases with the cube. The pressure and radial velocity are at
quadrature, so the motion is primarily incompressible (reactive). The velocities at the spherical surface are finite, but complex, to satisfy the force requirement. Viscosity has been neglected so surface slip is permitted. Little is gained by examining these variables further.

\[
p(r, \theta, t) = \frac{i3F_z}{4\pi r^2} \left( \frac{1}{1+ika} \right) \cos \theta e^{i(\omega t - k(r-a))}
\]
\[
u_r(r, \theta, t) = \frac{3F_z}{2\pi kr^3 Z_0} \left( \frac{1}{1+ika} \right) \cos \theta e^{i(\omega t - k(r-a))}
\]

4.3.3 Far Field

The relationships for the far sound field are

\[
p(r, \theta, t) = -\frac{3kF_z}{4\pi r} \left( \frac{1}{1+ika} \right) \cos \theta e^{i(\omega t - k(r-a))}
\]
\[
u_r(r, \theta, t) = -\frac{3kF_z}{4\pi rZ_0} \left( \frac{1}{1+ika} \right) \cos \theta e^{i(\omega t - k(r-a))}
\]

The phase angle of these variables varies with source size, due to the reference being source radius. See the value of bracketed term in the prior equations. The pressure and radial velocity are in phase, so the motion is primarily compressible (resistive). The influence of source size is discussed in the next section.

4.3.4 Source Size

The validity of the relationships given in {4.2} rests on the assumption that the source is a point. Comparing the above equations with those for the point source yields the requirement shown on the right. The source diameter must be less than 1/3 the wavelength; a specific criterion is given in {6.1.4}. When sound is created by fluid flows, source size is that of turbulent eddies which would satisfy this criterion in audible frequencies. Thus it is possible to make use of this approximation in many cases.

4.3.5 Radiation Impedance

The resistive and reactive impedances of the finite dipole are shown in Eqs. 4.14 below and in Figure 4-1.

\[
R_d = Z_0 \frac{k^4 a^4}{4 + k^4 a^4}
\]
\[
X_d = Z_0 \frac{ka(k^2 a^2 + 2)}{4 + k^2 a^4}
\]

At \(ka=1.41\) the resistance is 3 dB down and the reactance reaches its maximum value. The characteristic length is the radius, and the

Fig. 4-1. Finite dipole impedance.
characteristic speed is that of sound; the acoustical Strouhal number is 0.22. The relative impedances of the three source types are compared in Figure 6-1. The higher the order of source the less effective is the coupling at low frequencies. From a fluid mechanical viewpoint, the outflow on one side of the dipole at low frequencies sloshes to the other side. Much of the source energy is used to move the fluid incompressibly so the impedance is mostly reactive, not resistive. Both the radial intensity and the angular intensity have large components in the near field. The comparison to inductive circuits is appealing. The pressure fluctuation that accelerates the fluid incompressibly outweighs the component that compresses the fluid. The sound field is a minor part of the flow field near the source.

4.3.6 Sound Intensity

The radial and angular intensities are given in Eqs. 4.15. The angular intensity is purely reactive, while the radial has both components; the reactive decaying with distance more rapidly that the resistive. Note that in the far field, the intensity no longer increases continually with the square of the frequency; the finite size of the source creates an upper limit and that limit depends on source size. The value of the first (resistive) term in the radial intensity bracket is shown in Figure 4-2 for six source sizes in feet. The calculations were made for a distance of 100 feet; the frequency range was from 20 to 10000 Hz. It is clear that source size can play a significant role in determining the maximum value of intensity in the far field.

\[
I_r(r, \theta) = \frac{9 F_c^2}{16\pi^2 Z_0} \left[\frac{k^2 r^2}{r^4 (1 + k^2 a^2)} + i \frac{k^2 r^2 + 2}{kr (1 + k^2 a^2) r^4}\right] \cos^2 \theta
\]

\[
I_\theta(r, \theta) = \frac{i9 F_c^2}{32\pi^2 Z_0} \left[\frac{1 + k^2 r^2}{kr (1 + k^2 a^2) r^4}\right] \sin 2\theta
\]

(4.15)

Fig. 4-2. Dependence of radial intensity in the far field on source size.
4.3.7 Estimates of Sound Intensity

Direct intensity measurements, and their integral to sound power, require specialized equipment. How well does a sound pressure measurement approximate to those values? The relationship for this is

\[
EST[I_i] = \frac{p \cdot p^*}{Z_0} = \frac{9F_z^2}{16\pi^2 Z_0} \left[ \frac{1 + kr^2}{r^4} \right] \cos^2 \theta
\]

A mean square pressure measurement will represent the intensity correctly only in the far field. For a small source, it is

\[
L_p = 10 \log_{10} \left[ \frac{9k^2 F_z^2}{16\pi^2 r^2 Z_0 p_R^2} \cos^2 \theta \right]
\]

When measuring broadband dipole sound the \(kr\) value changes considerably so the problem of spectrum distortion occurs at low frequencies. The error is greater than 1 dB when \(kr\) is less than 2 as shown in Figure 4-3. It climbs rapidly for smaller values, so the error can be quite significant when very close to the source. To limit the measurement error to 1 dB, the distances in Table 4-1 must be met or exceeded for the lowest frequency to be measured. Fortunately, the distance restriction is not great.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>63</th>
<th>80</th>
<th>100</th>
<th>125</th>
<th>160</th>
<th>200</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inches</td>
<td>34</td>
<td>27</td>
<td>22</td>
<td>17</td>
<td>13</td>
<td>11</td>
<td>9</td>
</tr>
</tbody>
</table>

Fig. 4-3. Error in estimating intensity from pressure measurements.

Table 4-1. The minimum distance to avoid errors greater than 1 dB.

4.3.8 Sound Power

The sound power of a finite dipole is shown in Eq. 4.16. The upper limit of sound power caused by source size is clearly expressed in the equation. Note that the surface area of the sphere is \(4\pi a^2\). When the force is distributed over a large area (\(k^2a^2 \gg 1\)), the sound power is diminished.

\[
W_d = \frac{3F_z^2}{4\pi a^2 Z_0} \left[ \frac{k^2 a^2}{1 + k^2 a^2} \right]
\]
4.3.9 Dimensional Analysis

When the source size violates the point source model, the sound intensity and sound power are no longer proportional to the square of the frequency (Eqs. 4.15 and 4.16), so Eqs. 4.12 are no longer valid. Eqs. 4.17 reflect this change. There is no longer a Strouhal number dependence and the speed dependence changes to \( U^4 \) and the characteristic length \( L \) must be interpreted as source radius \( a \). This change in speed dependence due to source size is probably not common.

\[
W_d = \frac{3F^2}{4\pi a^2} = \frac{3\rho_0}{4\pi c_0} \hat{F}^2 U^4 L^2
\]

\[
\hat{W}_d = \frac{3}{4\pi} \hat{F} M
\]  

**Key Point:** When applied to a broad-band spectrum, only the high frequency components may meet this requirement so the spectrum contour changes from that at the source and the overall speed dependence changes {6.4.2}.

4.4 The Interaction of Two Single Frequency Finite Dipoles

Consider two dipoles separated by a distance \( 2h \). The geometry is the same as that in {3.4.1}. The phase of the velocity potential is defined at the surface of the first source.

\[
\phi_d(r, \theta, t) = \frac{3F_1}{4\pi r_1^2 k Z_0} (1 + ikr_1) \cos \alpha_i e^{i(\omega t - k(r_1 - a_1))} + \frac{3F_2}{4\pi r_2^2 k Z_0} (1 + ikr_2) \cos \alpha_j e^{i(\omega t - k(r_2 + a_2) + \delta)}
\]

\[
p(r, \theta, t) = \frac{i3F_1}{4\pi r^2} (1 + ikr_1) \cos \alpha_i e^{i(\omega t - k(r_1 - a_1))} + \frac{i3F_2}{4\pi r^2} (1 + ikr_2) \cos \alpha_j e^{i(\omega t - k(r_2 + a_2) + \delta)}
\]

\[
\frac{(1 + ikr_1)}{(1 + ika)} = \sqrt{1 + k^2 r_1^2} e^{i(\sigma_{r_1} - \sigma_\alpha)} = \beta_1 e^{i(\sigma_{r_1} - \sigma_\alpha)}
\]

\[
\frac{(1 + ikr_2)}{(1 + ika)} = \sqrt{1 + k^2 r_2^2} e^{i(\sigma_{r_2} - \sigma_\alpha)} = \beta_2 e^{i(\sigma_{r_2} - \sigma_\alpha)}
\]

\[
\tan \sigma_{r_1} = kr_1, \tan \sigma_{r_2} = kr_2, \tan \sigma_\alpha = ka
\]

The latter two equations apply when the source radii are equal. There are several variables that can create significant differences between the sources:

- The source radius (\( a_1 \) vs. \( a_2 \))
- The source strength (\( F_1 \) vs. \( F_2 \))
- The dipole axis (\( a_1 \) vs. \( a_2 \))
- The phase between the sources (\( \delta \))
- The distances (\( r_1 \) vs. \( r_2 \))

Each of these variables has been accommodated in **SoundSource**, but to retain some intuition about the sound field, one set of restrictions can be applied:

- The source sizes are equal (\( a_1 = a_2 = a \))

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The dipole axes are mirror symmetric \((\alpha_1=\eta-\theta, \alpha_2=\pi-\eta-\theta)\)

This particular restriction permits the geometry to be a dipole above a hard reflecting plane provided the phase angle is set to zero. The angle \(\eta\) is restricted to the range 0 to \(\pi/2\). When \(\eta\) is zero, one source points vertically upward, and the other points vertically downward. When \(\eta = \pi/2\), the axes of both sources are aligned horizontally. For the far field \(kr>>1\), \((\text{sound far field})\) and \(r>>2h\), \((\text{geometric far field})\), the approximations of Eq. 3.18 can be used. The mean square pressure becomes

\[
pp^* = \frac{9}{16\pi^2} \left[ \frac{F_1^2 \beta_1^2}{r_1^4} \cos^2 \alpha_1 + \frac{F_2^2 \beta_2^2}{r_2^4} \cos^2 \alpha_2 + \frac{2F_1 F_2 \beta_1 \beta_2}{r_1^2 r_2^2} \cos \alpha_1 \cos \alpha_2 \cos \left[ k (r_2 - r_1) + \sigma_1 - \sigma_2 - \delta \right] \right]
\]

\[
k (r_2 - r_1) = 2kh \cos \theta
\]

The second term is the relationship of the sources in the far geometric field. If the second source is absent, we recover the original equation. This equation requires too many words to explain, so it is presented in the SoundSource program. Both the orientation, phase and relative levels can be set. It is instructive to set the axes vertically, so there are two in-line forces acting in opposition to each other. Another is to set the axes horizontally and reverse the phase, so there are two parallel dipoles acting in opposition to each other. Later it will be shown that these two arrangements are quadrupole-like.

### 4.5 The Broadband Finite Dipole

For this case, it is necessary to work in the frequency domain. For outgoing waves the variables are

\[
\Phi(r, \theta, \omega) = \frac{3F_z(\omega)}{4\pi r^3 kZ_0} \left( \frac{1 + ikr}{1 + ika} \right) \cos \theta e^{-ik(r-a)}
\]

\[
p(r, \theta, \omega) = i\omega \rho_0 \Phi = \frac{i3F_z(\omega)}{4\pi r^3} \left( \frac{1 + ikr}{1 + ika} \right) \cos \theta e^{-ik(r-a)}
\]

\[
u_r(r, \theta, \omega) = \frac{\partial \Phi}{\partial r} = \frac{3F_z(\omega)}{4\pi kr^3 Z_0} \left[ \frac{2kr - i(k^2 r^2 - 2)}{(1 + ika)} \right] \cos \theta e^{-ik(r-a)}
\]

\[
I_r(r, \theta, \omega) = \frac{9F_z(\omega)}{16\pi^2 a^2 r^2 Z_0} \frac{k^2 a^2}{1 + k^2 a^2} \cos^2 \theta
\]

There is little new in these results when compared with the single frequency dipole. The far field pressure spectrum is distorted from the force spectrum by the radiation impedance term.

The sound power, the dimensionless force and the Strouhal number are now functions of frequency for a small source.

\[
W_d(\omega) = \frac{3\pi \rho_0}{c_0} \left[ \hat{F}^2(\omega) S^2(\omega) \right] U^3 L^2
\]

\[
\hat{W}_d(\omega) = 3\pi \left[ \hat{F}^2(\omega) S^2(\omega) \right] M^3
\]
The characteristic variables of length and speed must be those that characterize the Force and Strouhal spectra.

### 4.6 Two Interacting Random Finite Dipoles

For this case the variables are

\[
\Phi(r, \theta, t) = \frac{3F_1(\omega)}{4\pi r_1^2 kZ_0} \cos \alpha_1 e^{-ik(r_1-a)} + \frac{3F_2(\omega)}{4\pi r_2^2 kZ_0} \cos \alpha_2 e^{-ik(r_2-a)+\delta}
\]

\[
p(r, \theta, t) = \frac{i3F_1(\omega)}{4\pi r_1^2} \cos \alpha_1 e^{-ik(r_1-a)} + \frac{i3F_2(\omega)}{4\pi r_2^2} \cos \alpha_2 e^{-ik(r_2-a)+\delta}
\]

The two spectra can be different. The general case is sufficiently complex so that the equations must be modeled in SoundSource.

### 4.7 Fluid Mechanical Estimates of Dipole Directivity

The purpose of this section is to suggest that visualizing the oscillatory flow field close to a complex sound source might help to reveal the type of source through directivity characteristics without recourse to mathematics. Flow immediately around a dipole sound source in a free field is very similar to that around a bar magnet. Figure 4-4 shows a cross section of the streamlines near the source. The radial flow is obviously in the horizontal direction (compressible flow and sound), while there is none in the vertical direction (incompressible flow and no sound). This fits well with the results of \{4.3.6 and 4.3.7\}. An obvious omission in the theory is fluid viscosity which is needed to account for flow on the sphere surface. Would adding a viscous term help \{2.3.7\}?

![Fig. 4-4. Oscillatory streamlines around a dipole in free space.](image1)

![Fig. 4-5. Streamlines around a counter-rotating vortex pair.](image2)

As will be seen in \{4.9 and 4.10\}, many dipole sources are associated with unstable flows that contain vortices. Figure 4-5 shows the streamlines of the flow around a counter-rotating vortex pair; comparison is so close that dipole fields and vortex pairs are intimately related. The
weakness of the diagram is that the vortices do not reverse rotation in space to create an oscillatory flow as does the sphere. Time dependence results from vortex motion relative to a fixed position, as will be seen in later sections.

When a dipole approaches a plane surface, such as shown in Figure 4-6, the streamlines near the surface are compressed in the horizontal direction, predicting higher levels than would be found for the free dipole. The pressure gradient immediately above the plane attempts to drive all the flow through the gap, but as the surface is approached, not all of the flow can be recirculated as in a free field. Based on this, an estimate of sound field directivity would predict higher levels in the horizontal direction. Also, increased lateral velocities near the surface suggest increased oscillatory viscous shear layers on the surface, which is neglected in the theory.

A calculation of a horizontal dipole near a flat reflective surface was made and the results are displayed in Figure 4-7. The source was 1 inch in diameter, 12 inches from the surface at a frequency of 250 Hz. The lighter line is the directivity of a free dipole. The sound level in the horizontal direction is about 5 dB higher, supporting the fluid mechanical model. The decrease of level in the vertical direction results from some phase cancellation.

**Key points:** Modeling the hydrodynamic flow near a dipole sound source as streamlines provides estimates of sound field directivity without recourse to mathematics. The presence of moving vortex pairs is a strong indication that a sound source is likely to be a dipole. More complex geometries are discussed in later sections.

### 4.8 Modeling Dipoles

The basic feature of the theoretical dipole is a directional sound field resulting from a fluctuating force being applied in a specific direction. Some real sources may meet these requirements but others do not, so may be described only as dipole-like.

1. A fluctuating force in free space. It creates a sound field that has the characteristic cosine squared sound field of the theoretical model.
2. A fluctuating force acting near a reflective plane surface. The component of force perpendicular to the surface acts in opposition to the reflected force at the surface resulting in possible wave cancellation and directivity lobes. The component of force parallel to the surface is reinforced by the reflected force, increasing levels in that direction if distance from the surface is not too great. If the surface has finite impedance or the surface is curved, the directivity is further modified.

3. A fluctuating force acting at the edge of surface, such as an exhaust pipe or airfoil. This generally applies to flow over edges. The flow on either side of the edge may be coherent or incoherent producing various source strengths and frequency bandwidths. The flow laterally along the edge may be coherent or incoherent producing either a coherent line source or a number of independent (incoherent) smaller sources.

4. A fluctuating force acting in more complex geometries, such as a tire in motion. The number of dipoles may be large and may act in conjunction with other source types.

Although the word “force” is used exclusively above and in previous sections, forces can be tied to fluctuating momentum. Many of the situations described above apply to both Category I or II sources \{1.4\}, although it is not always possible to clearly separate the sources into the two categories. For example, the fluctuating flow around a cylinder \{4.10.1\}, both random and periodic, causes a sound field due to the fluctuating force exerted on the cylinder by the separated vortex street. At high speed the flow is sufficiently chaotic to fit into Category I. At lower speeds, the flow is more organized and would fit into Category II. For the purposes of this monograph, a source will be put into Category II whenever a strong tone predominates.

### 4.9 Modeling Category I Dipole Sources

Category I sources are those in which the generated sound is primarily a by-product of the source motion \{1.4.1\}. Although most of the mathematical models developed in this chapter are approximations to reality, it is still possible to learn much about a sound source using these models, along with the scaling rules and judicious choice of characteristic variables.

#### 4.9.1 Unbaffled Loudspeaker

Audio professionals all design baffles around loudspeakers to enhance the low frequency response of the system. At times, the baffle is supplemented by a resonant cavity to further boost low frequencies. Essentially, they are converting a dipole source to a monopole-like source.

Figure 4-8 shows the degradation in output caused by an eight inch diameter speaker being unbaffled instead of baffled. There is a significant reduction in audio frequencies less than 1000 Hz. In
addition, the directivity of a baffled speaker is nearly that of a piston in a plane surface \{3.10.1\},
while the unbaffled speaker still has the cosine squared directivity pattern \{4.3\}.

**Key Points:** The act of baffling a speaker creates two changes. The change from a dipole to a
monopole-like source increases radiation efficiency at low frequencies. The sound field
directivity changes from a null in the plane at right angles to the speaker axis to a more uniform
directivity pattern.

### 4.9.2 Airfoil Sound

There are several sound sources from an airfoil in flight, all related to lift and drag
fluctuations. This discussion is restricted to flight sufficiently subsonic that shock waves do not
form on the upper surface. The first source is due to the encounter with large scale variations in
the atmosphere, resulting in fluctuations of the angle of attack \{4.9.2.1\}. Another source is due
to the turbulent trailing edge flow \{4.9.2.2\}, and the last is due to trailing edge flow that has
nearly pure tones associated with it \{4.10.1\}.

#### 4.9.2.1 Angle of Attack Changes

Consider the case where an airfoil is immersed in a fluid stream of speed, \(U\), at an angle
of attack \(\alpha\). The airfoil does not change direction with respect to the earth, but rather, the large
scale turbulent eddies create local angle of attack changes as the airfoil passes through them.
Free atmospheric turbulence scales are on the order of meters. If the highest frequency of
interest is such that the chord \(C_h\) and airfoil length \(L\) are less than one-sixth of a wavelength
(\(kL < 1\)), the point dipole model is a reasonable approximation. This approach may have some
value only for small aircraft. For large planes, and gliders with a high aspect ratio
a distribution
of sources along the airfoil requires only that the chord be smaller than one-sixth wavelength.

Typical equations for the lift and drag of an airfoil are given in Eq. 4.22 below.

\[
C_l = \frac{2F_l}{\rho U^2 ChL} = K(\alpha - \alpha_0) \tag{4.22}
\]
\[
C_d = \frac{2F_d}{\rho U^2 ChL} = a + \frac{K^2}{b}(\alpha - \alpha_0)^2
\]

The \(\alpha_0\) is the angle of attack for which the lift force is zero. Data from NACA airfoil
2412, for example, suggests that the lift coefficient is nearly linear in a range from \(-8\) to \(+15\)
degrees, with the constant \(K=0.1\). Zero lift was at \(-2\) degrees. For the nonlinear drag
coefficient, the frictional parasitic drag factor \(a\) was 0.007, and the lift factor \(b\) was 300. The
airfoil area can be expressed in two ways, depending on whether the entire airfoil is involved or
not. The sound power equation, derived from Eq. 4.7, can be expressed for both the lift
(subscript \(l\)) and drag (subscript \(d\)) dipoles as

\[
W_l = \frac{3}{4\pi Z_0 c_0^2} \left[ \frac{\partial F_l}{\partial t} \right]^2
\]
\[
W_d = \frac{3}{4\pi Z_0 c_0^2} \left[ \frac{\partial F_d}{\partial t} \right]^2
\]

\[
W_l = \frac{3Z_0 M^4 C_h^2 L^2}{4\pi} \left( \frac{K^2}{4} \right) \left[ \frac{\partial \alpha}{\partial t} \right]^2
\]
\[
W_d = \frac{3Z_0 M^4 C_h^2 L^2}{4\pi} \left( \frac{K^4}{b^2} \right) \left( \alpha - \alpha_0 \right)^2 \left[ \frac{\partial \alpha}{\partial t} \right]^2
\]

\(\alpha - \alpha_0 = \alpha_1 + \alpha_2 \sin \omega t\)
The lift and drag forces have been converted to expressions in terms of the angle of attack. For sinusoidal changes in angle of attack the last equation of Eqs. 4.23 may be substituted into the earlier equations. The drag dipole, being nonlinear, has a complex amplitude structure and the sound power is about 63 dB down from the lift dipole. Consider a hypothetical situation where a small aircraft, with two wings of two foot chord and twelve foot wingspan, is penetrating 20 foot diameter eddies at 150 knots whose magnitude results in one degree angle of attack changes. Lift fluctuations would generate about 24 acoustic Watts at 12 Hz. Without specific information on the atmospheric structure, realistic numbers are difficult to determine. **Key Points:** Penetration of atmospheric turbulence by an aircraft can result in significant sound but at very low frequencies. Because of the low frequency it is generally ignored. The characteristic speed is that of the aircraft. The characteristic length is the wind chord. The role of the wing length as another characteristic length is yet to be defined; if the scale of atmospheric turbulence is small, it defines a multiplicity of independent sources, while if the scale is large it defines the total force applied to the medium as a dipole. Each wing is likely to be a partially independent source.

### 4.9.2.2 Trailing Edge Sound

The turbulent flow over an airfoil results in sound generation, particularly at the trailing edge. It is enhanced when flaps are deployed. If the flow is laminar, hydrodynamic feedback can create a Category II source \(4.10.2\). A similar process can occur near the leading edge if flaps are deployed there (it is unlikely that the scale of atmospheric turbulent eddies are sufficiently small to create significant leading edge sound). Since it is difficult to measure in flight, laboratory tests have shown its nature. Figure 4-9 [29] shows an example of such a test. A boundary layer flow was run on either side of a flat plate past a square-edged trailing edge. Note the nearly pure tone at 2000 Hz with a Strouhal number of 0.21 \(4.10.2\). Once again the magic number of Strouhal appears. The characteristic speed was the mean speed of the jet, and the characteristic length was chosen as the trailing edge thickness. The better characteristic length would have been the boundary layer thickness, but fortunately the two dimensions were almost the same. The concept of boundary layers is attributed to Ludwig Prandtl (1875-1953). Very little of his work was known to those creating the early aircraft industry. Further experiments [5] used both single and double-sided boundary layers flowing over a finite length plate. Rectangular
nozzles were used to create the flow. The measured sound field was clearly dipole-like (modified slightly by the plate presence). Figure 4-10 shows sound spectra that have been scaled using the $U^6$ power law over a speed range of 3 to 1. The frequencies shown in the figure are actually frequencies that have been Strouhal scaled. The fit to Eq. 4.12 is quite good, but not exact. The slight discrepancy in level and frequency suggests that both the dimensionless force and the Strouhal number had weak dependence on the Reynolds number.

For these experiments, the plate width was greater than the lateral dimension of the jets. On an airfoil, the above concept would lend itself to a sum of independent dipoles arrayed laterally; the spacing being the lateral correlation scale.

Since the dipole sound field is based on the time rate of change of the force, reduction of sound must be accomplished by reducing that rate. One possible means would be for the opposite sides to gradually sense each other prior to the trailing edge, reducing the rate at the edge. This can be done by a section of graduated porous or flexible materials. This concept was tested on a propeller fan with porous blades with some success [30].

**Key Point:** Dynamic similarity for a dipole sound source works well on trailing edge noise.

### 4.9.3 Propeller Sound

The motion of some aircraft depends on the propulsive force generated by the lift of propellers. Because the propeller is rotating there is a rotating static lift force, a dynamic lift and drag force due to turbulent flow, and a blade thickness effect. Detailed analyses require integration of a distribution of sources on each blade. Theory and experiment are well developed [31]. A much simpler approximate analysis can be made by presuming a sum of point sources, either one centralized force on each blade or several along its length as suggested in Figure 4-11. Again the number of sources is determined by the correlation lengths along the blade.

Figure 4-12 shows the relationship between the thrust, lift, and drag forces to the lift and drag dipole sound sources. It is based on propeller angle of attack which varies with radius. The two dipole forces are related to the thrust by $T = F_l \cos \alpha - F_d \sin \alpha$. $F_l$ is the lift dipole source strength and $F_d$ is the drag dipole source strength. The following sub-sections look at the three types of sound sources on subsonic propellers, with the intention of indicating simple approaches to estimating the sound power from these devices, and defining the two characteristic variables. To simplify, the following discussion is in a frame of reference moving with the propeller, so that the time delays to a fixed frame of reference can be ignored; they do not contribute to understanding the phenomenon.
4.9.3.1 Random Sound

This sound is primarily the sound caused by turbulent flow over the trailing edge of the blades \{4.9.2.2\}. When the prop pitch is correct, the boundary layer remains attached, otherwise flow separation occurs and large changes in the spectrum occur. This difference can be heard when the operator of a variable pitch aircraft fails to change pitch after takeoff. Considering that there is a distribution of dipole sources on each propeller, such as suggested in Figure 4-11, Eqs. 4.12 can be expanded to show the local sound power as a function of radial position for a specific frequency. It is shown in Eqs. 4.24.

\[ W(r) = \frac{3\pi \rho_0}{c_0^3} \left( \frac{f \delta(r)}{r \omega_0} \right)^2 \left( \frac{F}{\rho_0 r^2 \omega_0^2 A(r)} \right)^2 (r \omega_0)^6 A(r) \]  

\[ U = r \omega_0 = \frac{2\pi r \text{ RPM}}{60} \]

\[ A(r) \approx \beta \delta(r) Ch(r) \]  

(4.24)

This model could be expanded to a broadband spectrum \{4.5\}, but it is not more illuminating. Look first at frequency. Based on \{4.9.2.2\}, the characteristic length for frequency is the boundary layer thickness \(\delta\), which defines eddy size. It will be a decreasing function of radius. The frequency spectrum maximum would increase with blade speed \(U\) and would also increase as the boundary layer thins with increasing radius. Look at the dipole source strength. It would increase with \(U^2\). The effective radiation area \(A\) would be some fraction of the blade chord, and the radial correlation length related to the boundary layer thickness; the fraction \(\beta\) is a constant of proportionality. As the radius increases, both the boundary thickness and the chord decreases, suggesting that although the overall speed dependence of the sound power at any radius appears to follow the sixth power law, the change in the other dimensionless variables with radius implies that the overall sound power may not follow that law exactly. The discussion tacitly assumes that the propeller does not encounter the wake of the previous blade, such as it might in a helicopter. The structure of the boundary layer over the blade is altered considerably in that case.

**Key Points:** A simple dipole model using appropriate characteristic variables provides a nice knowledge framework around sound from a propeller blade (Eqs. 4.24). Knowing the blade shape, both chord length and angle of attack, and the boundary layer thickness as a function of radius it would be possible to estimate the sound power and spectrum contributions at each radius. The radial eddy correlation lengths are not known, but some insight might be obtained by modeling with a variable the number of independent sources.

4.9.3.2 Steady Lift and Drag Sound

Since the steady lift force is almost oriented with the thrust vector, it may be considered as a rotating fixed force on the blade oriented along the propeller axis (with a slight wobble). The steady drag force is more nearly oriented with the plane of the propeller, so may be considered a rotating fixed force on the blade oriented tangential to the propeller axis (again with a slight wobble). Time dependence is due to changes in distance from the observation point. Thus the sound pressure of steady lift and drag force measured at a distance would vary most in the plane of the propeller, since the distance variation is greatest there. See \{4.10.4\} for an example of the frequency spectrum of a dipole created by a rotating fixed force. Since the frequency is a multiple of the RPM, it is a multiple of 37 Hz for a small aircraft at 2200 RPM.
Key Point: The steady lift and drag forces on a propeller create sound, but because of the low frequencies it is often neglected.

4.9.3.3 Thickness Sound

As air passes over a propeller blade, it must separate at the leading edge and then merge after the trailing edge. At a particular point in space, the separation creates a monopole source whose strength is related to the blade thickness. NACA 2412 airfoil was used to estimate the wave form of the source. The results are shown in Figure 4-13 for a four bladed propeller with a four foot blade length and a 4 inch chord. The graph shows the displacement waveform as a fraction of one-quarter cycle. Close to the hub the volume change is a larger fraction of the period, while at a larger radius it is considerably smaller. It is clear that the source is more like a periodic pulse than a sinusoid. Fourier analysis would suggest that the harmonic content of the source at greater radii is much richer than closer in. For a four bladed propeller rotating at 2200 RPM, the fundamental frequency is 147 Hz. The Helmholtz number, \(ka\) is near 0.1, where \(a\) is about half the largest chord. As a result, it is possible to model the action as one monopole source whose volume is the total volume of the blade. Eq. 3.10 may be used to make an approximate estimate of sound power at the fundamental frequency, presuming the chord is constant to the blade tip and the sound field is radially symmetric. A calculation would show the sound power to be about 0.0015 Watts and about 44 dB at 50 feet. The improved radiation efficiency of the higher harmonics may offset the amplitude reduction with frequency, so higher levels may occur.

Key Points: Making use of the physics of blade passage permits the source type to be defined and thereby estimates of sound power to be made. The characteristic length is the chord which acts to define the oscillatory volumetric flow at any radius. The characteristic speed is the blade speed at any radius.

4.9.4 Bicycle Spoke Sound

The air flowing over a bicyclist and his bicycle creates sound. Potential sources are those caused by vortex shedding behind the rider, behind the frame, behind the spokes, and the tire interaction with the road. The sound from spokes is likely to be loudest at mid frequencies since the speed at the wheel top is twice the bike speed and so the frequency will be higher than that from the other sources. The center of the wheel moves at the speed of the vehicle \(U_0\). Most spokes are cylindrical in cross section, of diameter \(d\), so the concepts of the Aeolian tone apply.
Because the velocity varies strongly along the length of the spoke, it is highly likely that the correlation length $\Delta r$ along the spoke is small, perhaps similar to lateral correlations on trailing edge noise \cite{4.9.2}. Figure 4-14 shows a wheel of radius $R$. The speed of each part of a spoke is determined by the position $r$ radially along the spoke and the angular position of the spoke in the rotation. Applying Eqs. 4.12 locally, results in Eqs. 4.25.

\[
U(r) = U_0 \left[ 1 + \frac{r}{R} \cos \delta \right]
\]

\[
W(r) = \frac{3\pi \rho_0}{c_0^3} \left( \frac{f(r)}{U(r)} \right)^2 \left( \frac{F}{\rho_0 U^2(r) d \Delta r} \right)^2 U^0(r) d \Delta r
\]

\[
f(r) \approx \frac{0.2U_0}{d} \left[ 1 + \frac{r}{R} \cos \delta \right]
\]

The velocity function must certainly have been used to calculate the drag induced by spokes on racing bikes, but here the interest is in sound output. The spoke diameter $d$ is generally close to 1/16 inch while spoke length $R$ is near 12 inches. The correlation length $\Delta r$ is unknown. At a bike speed of 25 mph, the Reynolds number will vary from 0 to 2200, so the lower spokes will have laminar flow and a purer tone, while the upper spokes will have turbulent flow and a broader band spectrum. Using the dynamic similarity rules permits developing useful information about spoke sound. Figure 4-15 shows the relative sound power level of a spoke as it rotates from the top of the wheel to the bottom. The solid curve is for the part of the spoke close to the rim, while the dashed line is at half radius. The dipole speed dependence guarantees that most of the sound is created while the spoke is at the top of the wheel. Figure 4-16 shows the frequency of the tone (or of the band maximum) for the same conditions as in Figure 4-15. It is clear that the sound spectrum is dominated by high level-high frequency sound at the rim. Anyone listening to bicycles in a high speed race can confirm this comment.

A computer model of spoke sound was created in SoundSource. The spectrum slope above the maximum was modeled on trailing edge noise spectra. Twelve independent radiating lengths along the spoke ($\Delta r$) were chosen for each of the 36 spokes in the wheel. The frequency spectrum was calculated for each of the 432 sources based on Strouhal modeling and a speed of
15 mph. The source strength was arbitrary so the results shown in Figure 4-17 are representative of the one-third octave band spectrum contour, but not necessarily the actual level.

Some sound measurements were made of bicycle race videos and one of a bicycle passby; they are shown in Figure 4-18. Since the videos were uncalibrated, the levels were adjusted to match; the spectrum contour being of primary interest. The evident low frequency sound is from the frame and rider; sprocket wheel sound is likely the left peak. The passby speed was 15 mph and the video samples were of a group of riders on a flat surface, so the speed of each person was likely the same. Comparing the spectra in the two figures suggests that the dipole model is a reasonable method for estimating the spoke contribution to bicycle sound.

**Key Points:** Adding the aeolian tone concept to a simple dipole model permits reasonable estimates of sound power and spectra to be made when data are available. The characteristic length is the spoke diameter. The characteristic speed is that of the bicycle to which the actual speeds can be tied explicitly (Eqs. 4.24).
4.9.5 Exhaust Flow Sound (Radial Dipole)

As the exhaust speed increases, the volumetric flow rate fluctuations \{3.10.4\} are supplemented by highly turbulent boundary layers. Boundary layer thickness is the characteristic length which determines eddy sizes. Since the correlation between turbulent eddies is determined by their size which is much less than the perimeter, so it is possible to model the exhaust flow sound source as a number of independent sources as suggested in Figure 4-19. This results in a circular array of trailing edge noise. The geometry of \{4.9.2.2\} is wrapped into a circle, creating what may be called radial dipoles, whose axes are directed radially outward in the exit plane. A classical instance of this source type is the sound created during acceleration of a recreation vehicle with a large engine and an undersized exhaust pipe. Dipole sound should follow the \(U^6\) law as opposed to the \(U^4\) law for volumetric flow rate fluctuations, so this source will become dominant above a certain mean speed \{6.4.4\}.

To model the source, the integral is replaced by a sum of independent dipole sources. The relevant equations are

\[
pp^* = \frac{9k^2F_r}{16\pi} \sum_{n=1}^{M} \frac{\cos^2 \theta_n}{r_n^2}
\]

\[
\cos \theta_n = \frac{r \sin \theta_n \cos \alpha - R_0}{r_n}
\]

\[
r_n^2 = R_0^2 + r^2 - 2rR_0 \sin \theta \cos \alpha
\]

The coordinate system is shown in Figure 4-20. The distance \(r\) is from the center of the exhaust plane to the measurement point and \(\theta\) is the angle from the exhaust axis. The distance from the actual source to the measurement point for the \(n\)th source is \(r_n\) and the angle from that source axis is \(\theta_n\). \(R_0\) is the radius of the pipe and \(\alpha\) is the angle around the periphery, zero being in the measurement plane and toward the measurement point. Again, the unknown variable is the correlation length. Based on the trailing edge sound results \{4.9.2.2\}, the characteristic length should be the boundary layer thickness \(\delta\) and the characteristic speed should be that of the mean flow \(U\).

A model was created in which there were 36 independent (uncorrelated) sources around the periphery (the index \(n\) in Eq. 4.26), so the correlation length was at each 10 degrees or 0.175 \(R_0\). The sound pressure field in a quadrant was modeled in SoundSource. Figure 4-21 is an example of the results for a pipe 2 inches in diameter radiating with the broad-band spectrum shown in the figure; the distance was 20 feet. Because the dipole is removed from the pipe axis,
there is no null as would occur with a point dipole. The three dimensional directivity is a cosine-squared donut. No lobe patterns arise, even from single frequency sources.

There is a second type of motion that can give rise to a radial dipole. A highly oscillatory volumetric flow rate (as suggested in Figure 3-9) gives rise to a ring vortex. Examples would be the pulse of a V1 {3.11.3}, the flatulence of a Harley-Davidson motorcycle {3.11.5}, or the blowing of smoke rings. The pressure fluctuation driving the flow at the exit plane results in a monopole field {3.2}, but that pressure is different than that immediately on the far side of the lip outside the exit. There are two significant differences from the previous example: the spectrum is that of the volumetric flow rate fluctuations which may be a single frequency and all the sources around the periphery are coherent (in phase). Volumetric flow fluctuations would be at a lower frequency than that associated with boundary layer sound. If the sound power of the dipole is compared with that of the monopole, by using the driving pressure for the dipole as that for the monopole source, there results a simple relationship between the two:

\[
\frac{W_d}{W_m} = \frac{3\left(\frac{A_d}{A_m}\right)^2}{4} (ka)^2
\]

\[ (4.27) \]

Fig. 4-19. Directivity of the broad-band radial dipole.

If the area on which the dipole force acts is the pipe periphery of radius \( a \) and some unspecified, but short, distance \( b \) down the pipe and the area of the monopole source is that of the pipe exit, the area ratio above becomes \( b/a \). Typically, this is a small number. As an example, the fundamental exhaust frequency was chosen to be 125 Hz, near that of a Harley motorcycle or V1. With a radius of near one inch, the dipole sound power about 32 dB down from the monopole sound power; certainly buried in the monopole sound and hard to detect.

**Key Points:** The dipole sound emitted from an exhaust pipe can come in two forms. The first is that of turbulent boundary layer flow. The maximum sound level is in the plane of pipe exit. The characteristic variables are the boundary layer thickness and the mean speed of the flow. The second is associated with a strongly varying mass flow. One component of the sound field is like that of a monopole {3.10.4} while another, possibly much weaker component, is that of a radial dipole. The characteristic variables are pipe radius and mean speed. Since there are two source types with differing speed dependences and sound field directivity can vary, changes of mean speed should be expected to result in changes in the speed exponent and directivity. Once again, the model shows the importance of defining the correlation length in order to develop useful estimates of sound output.
4.9.6 Automobile Tire Sound

The monopole aspects of tire sound were considered in {3.11.6}. Consider the dipole aspects. When the tread first encounters the road, the tread is deformed and the air at the tread surface facing the direction of travel must be expelled. Similarly when the tread leaves the road, the tread deformation is released, and air at the rear must fill the gap which is left. It is likely that the deformation does not change the center of mass of each tread. Another unlikely possibility is if two adjacent treads deformed out of phase. Both cases suggest that dipole sound is unlikely. Tread deformation implies a quadrupole aspect {5.1}, but at typical vehicle speeds is unlikely to be important.

Tire squeal occurs when tire direction is changed abruptly or the vehicle is rapidly accelerated. It is likely a stick-slip interaction that causes deformation of the tread and consequent tread oscillation. All three source types are likely to be evident. Air pumping monopoles would exist if the correlation between treads was proper. Lateral tread deformation results in an oscillatory change in the center of tread mass and thus a dipole source. Tread deformation can also result in quadrupole sources, but relevant speeds suggest that their contribution is likely to be small. There appears to be no study to determine the relative contribution of each source type.

**Key Points:** If appears that dipole sound from automobile tires is insignificant relative to monopole sound monopole sound {3.11.6} under normal driving conditions. If dipole sources were present, the characteristic variables would be tread size and vehicle speed. Tire squeal results in a more complex array of sound sources.

4.10 Modeling Category II Dipole Sources

Category II sources are those in which the generated sound has a strong impact on the source {1.5.2}. It is possible to learn much about these sources using theoretical models along with the scaling rules and judicious choice of the characteristic variables. The examples below are restricted to those in which the feedback is sufficiently controlling that a single, or nearly single, frequency results. In many, but not all, cases, there is a coupled resonant structure. In many mechanical devices, resonant structures are common, either deliberately (musical instruments) or accidently (duct branches or wheel wells).

4.10.1 Flow around Objects

There are several interesting situations in which sound is generated by flow around a fixed object. In each case the wake behind the object generates finite vortices that feed back and cause a fluctuating momentum around the object, which then results in a dipole sound source. In certain Reynolds number ranges, the feedback is hydrodynamic (Class I) and periodic, resulting in nearly pure tones. Several examples are given below.

4.10.1.1 Cylinder Sound (Aeolian Tone)

The steady flow over a cylinder (or similar object) placed across the flow, will generate vortex shedding and consequent sound. The early Greeks used this phenomenon to develop the Aeolian harp and the sound was called an aeolian tone after Aeolus, God of the Wind. Whistling telephone wires, automobile radio antennae, and certain automobile front grilles are other
examples of this tone. At very low Reynolds numbers the flow is stable, forming two fixed vortices. As the speed is increased, the flow, although laminar, becomes unstable and vortices are shed alternately. The vorticity (hydrodynamic) feedback influences the formation of new vortices and exerts a fluctuating force on the cylinder. The flow field is shown in the famous photograph by Gary Koopman (Figure 4-20). Theodor von Karman (1881-1963) first identified and analyzed the flow behind objects like a cylinder and since then this special flow has been called the Karman vortex street. Vincenz Strouhal (1850-1922) was the first to scientifically investigate the sound emitted by flow around a rigid cylinder. At low Reynolds numbers the tone was pure and the frequency was proportional to the steady flow speed $U$ and inversely proportional to the cylinder diameter $d$. For a range of speeds and diameters he developed the relationship shown in the first equation of Eqs. 4.28; this is the origin of the (fluid mechanical) Strouhal number. It is surprising how often oscillatory flow phenomenon have a Strouhal number near this value. The characteristic length is the dimension of the object lateral to the flow and the characteristic speed is that of the impinging flow. At higher flow speeds ($Re>5000$), the tone is no longer discrete but the spectrum has a peak centered near $St=0.2$ over a broad range of Reynolds numbers up to $2\times10^7$. Figure 4-21 shows an example spectrum. Evidence suggests that the Strouhal number is a weak negative function of Reynolds number as seen in the second of Eqs. 4.28. This suggests that the dynamic similarity approximation is reasonable.

The fluctuating force exerted on the cylinder is a result of the flow circulation around it caused by the alternate vortex separation as suggested in Figure 4-22. The fact that the vortices are not directly behind the cylinder suggests that the force vector has both a lift and drag component resulting in lift and drag dipoles. An approximate way to relate the sound generated to the flow characteristics is to perturb the standard drag equation (lift measurements for cylinders are generally not available). The upper equations of Eqs. 4.29 relate the dimensionless lift and drag forces to the perturbation variables. The lower equations make use of Eqs. 4.12 to develop an expression

$$St = 0.2 = \frac{fd}{U} \quad (4.28)$$

$$St = 0.198 \left( 1 - \frac{19.7}{Re} \right)$$

![Fig. 4-20. Vortex street behind a cylinder.](image1)

![Fig. 4-21. Sound spectrum of turbulent flow over a cylinder.](image2)
for the lift and drag sound power. The cylinder diameter is $d$ and $w$ is the cylinder length. The fluctuating speed in the stream direction is $u'$ and the lateral fluctuation is $v'$.

$$F = F_d + F_d' + F_l = \frac{C_d \rho_0}{2} \left( \frac{U + u' + v'}{U} \right)^2 dw$$

$$\hat{F}_d^2 = \left( \frac{F_d'}{\rho_0 U^2 dw} \right)^2 \approx \left( \frac{C_d u'}{U} \right)^2$$

$$\hat{F}_l^2 = \left( \frac{F_l'}{\rho_0 U^2 dw} \right)^2 \approx \left( \frac{C_d v'}{U} \right)^2$$

$$W_d = \frac{3 \pi \rho_0}{c_0^3} S^2 U^6 dw C_d^2 \left( \frac{u}{U} \right)^2$$

$$W_l = \frac{3 \pi \rho_0}{c_0^3} S^2 U^6 dw C_d^2 \left( \frac{v}{U} \right)^2$$

The sound output can be estimated from knowledge of the turbulence intensity. The magnitude of velocity fluctuation is difficult to assess. If details of vortex development were known, the circulation might be calculated using the methods of inviscid fluid mechanics. Each time a vortex is shed, the drag velocity fluctuation $u'$ has the same sign, but the lateral velocity fluctuation $v'$, has opposite signs, since the vortex is shed on alternate sides, so the drag dipole is expected to be twice the frequency of the lift dipole. Phillips [32] suggested that at low Reynolds numbers the tone is reasonably pure and the dimensionless force can be expressed as

$$\hat{F}_d = 0.04 \cos 2 \omega t$$

$$\left( \frac{u}{U} \right)^2 = 2.5 \times 10^{-4}$$

$$\hat{F}_l = 0.38 \cos \omega t$$

$$\left( \frac{v}{U} \right)^2 = 2.2 \times 10^{-2}$$

Since the drag coefficient is near 1.8 for a cylinder, we can deduce that turbulence values are those shown. The lateral velocity fluctuations are two orders of magnitude greater than the longitudinal, and the lift dipole is 20 dB above the drag dipole. The prediction of a drag dipole from Figure 4-22 was confirmed by Phillips measurements.

The influence of width is not well known. When the flow is laminar and uniform in the width direction it is likely that the flow is well correlated over the entire width. At higher speeds, the lateral correlation length will be less than the cylinder width so sections of the cylinder will radiate as independent sources.

Key Points: The study of flow over a cylinder has led to some fundamental concepts of dynamic similarity and how the equations for fluid motion can be used to estimate sound generation. The characteristic length is the cylinder diameter and the characteristic speed is that of the mean flow.
A second characteristic length is the lateral correlation length for turbulent flow. The source type was clearly defined as a dipole.

**4.10.1.2 Flowmeters**

Of what use is knowledge of Aeolian tones, aside from making a musical instrument? For one thing, there are now several accurate flow meters that are designed to take advantage of the constancy of the Strouhal number with Reynolds number to provide a linear relationship between flow rate and meter frequency. They are called vortex meters. A particular shaped object is placed within a pipe and a pressure sensor is embedded either in the pipe wall or in the inserted object. Although a number of shapes have been used, there are several that work well. Figure 4-23 is an example of one called the *Deltameter*, since the shape is that of a trapezoid; the wider end faces upstream. The data suggest a nearly linear relationship with flow speed over nearly a 1000 to 1 range of Reynolds numbers (12,530 to 1,181,000). Orifice plates typically have a 5 to 1 range, while turbine meters may have up to 100 to 1. It should be noted that three dimensional (viscous) effects occur at low Reynolds numbers so dynamic similarity is not achieved there. The dependence of Strouhal number on Reynolds number for this confined geometry is slightly negative as was found also in {4.10.1.1}.

**Key Point:** The confinement of an object within a circular pipe still permits the concepts associated with flow over a cylinder to be useful. The characteristic length and speed was well defined so the range of adherence to dynamic similarity could be determined.

**4.10.1.3 Other Objects.**

Photographs taken from space have shown alternate arrangements of clouds around mountains; Figure 4-24 is one example. Does this type of event create sound? The NOAA laboratories in Boulder, Colorado, detected an extremely low frequency sound, and by triangulation were able to determine it was vortex shedding from a volcanic cone in the Aleutian chain, so the answer is definitely yes.

Another aspect is the response of the object to the applied force. Although not cylindrical in shape, the elasticity of the Tacoma Narrows Bridge responded to the vortex shedding until it failed. Galloping telephone wires are yet another example of vortex shedding.
example; line dampers are used at anti-node points of the wire to reduce destructive motion. There are numerous other examples. Does the motion of the object in response to the force enhance sound generation?

4.10.2 Airfoil Trailing Edge Tone

The sound from sailplanes has been measured. At low speeds the boundary layer is laminar and vortex shedding similar to that of a cylinder occurs but at a trailing edge \{4.9.2.2\}. The result is a nearly pure tone. It is clear that a fluctuating force is exerted at the trailing edge resulting in a dipole sound field.

Figure 4-25 shows a one-third octave band spectrum taken at a sailplane flyover. The airspeed was 51 m/sec (157 ft/sec), the chord was 17 inches, and the frequency was near 1400 Hz. Based on a Strouhal number of 0.20 \{4.10.1.1\}, the characteristic length can be calculated to be near ¼ inch; the boundary layer thickness. **Key Point:** The approximate thickness of the boundary layer can be estimated from sound measurements using dynamic similarity rules. Since boundary layer thicknesses are reasonably known, air speeds might be estimated from the sound frequency.

4.10.3 The Ring Tone (Ring Dipole)

The flow from a circular orifice impinging on a toroidal ring of the same diameter as the orifice will result in a tone; it is called a ring tone. Small disturbances at the ring feed back to the orifice to be amplified by the flow instability. The unstable flow results in a set of symmetric (ring) vortices that later impinge on the ring. The passage of a vortex is shown schematically in Figure 4-26 in three steps. The vortex creates a circular dipole flow field whose axis varies as the vortex passes. The fluid mechanical diagram suggests that the main axis of the force on the ring is in the direction of the jet flow resulting in a ring dipole where all components of the force around the ring are in phase. It also suggests that there is a lateral component of force which can only be interpreted as a radial dipole. Unlike that described in \{4.9.5\}, all components of the force around the ring are in
phase. The flow vectors in the figure are merely suggestive of direction. When two vortices are equidistant from the ring, one being beyond and the other approaching, the net circulation around the ring is zero; the null point for the dipole flow oscillation. The diagram shows the vortices passing through the interior of the ring; if they passed outside, the results would be the same.

For a ring of radius \( R_0 \), the orientation of the main dipole axis is perpendicular to the plane of the ring as shown in Figure 4-27. The coordinates from the center of the ring are \( r \) and \( \theta \), while \( \alpha \) is the angle in the plane of the ring between the source on the ring and the measurement direction. Eqs. 4.31 show estimates of the sound intensity for the ring by a summation of \( m \) in-phase sources. The phase at the measurement point of each small section of the ring is compared with that of the zeroth term \( (\alpha=0) \).

\[
\frac{pp^*}{Z_0} = \frac{9k^2F^2}{16\pi^2Z_0} \left[ \cos^2 \theta_0 + \sum_{n=1}^{m} \cos^2 \frac{\theta_0}{r_n^2} + 2\cos \theta_0 \cos \frac{\theta_0}{r_n^2} \cos \left(k(r_n - r)\right) \right]
\]

\[
\cos \theta_n = \frac{r \cos \theta}{r_n}, 0 < n < m
\]

\[
r_n^2 = R_0^2 + r^2 - 2rr_0 \sin \theta \cos \alpha
\]

Fig. 4-27. Ring tone geometry.

Experiments have been performed on the ring tone [21]. Figure 4-28 shows the relationship of frequency to Reynolds number. If the Strouhal number were graphed in lieu of the frequency, it would have shown that contours were reasonably constant. Close examination of the data in the figure showed a slight dependence of Strouhal number on Reynolds number. The results are similar to those for the hole tone \{3.12.1\}. Sound field measurements, shown in
Figure 4-29, clearly indicate that the sound source was a dipole. Since there were no reflecting surfaces near the source, the rise of level in the plane of the ring suggests that the radial dipole component existed.

The dipole model can be used to estimate the sound power of the axial component of the ring tone. There are two characteristic lengths in this case: the diameter of the ring material \(d\) and the radius of the ring \(R_0\). The frequency is related to \(d\), while the dimensionless force is related to \(d\) and \(R_0\). The characteristic speed is that impinging on the ring, which is some fraction \(\beta\) of the center line speed \(U_0\). The parameter \(\beta\) depends on ring diameter and ring distance from the orifice, while the centerline speed depends on the ring distance from the orifice. Eq. 4.32 shows the alternative form of Eqs. 4.12.

\[
W_d = \frac{6\pi^2 \rho_0}{c_0^3} \left[ \frac{F_z}{2\pi \rho_0 \beta^2 U_0^2 R_0 d} \right]^2 \left[ \frac{f d}{\beta U_0} \right]^2 (\beta U_0) R_0 d
\]  

(4.32)

**Key Points:** Although flows impinging on fixed objects generally create broadband sound, special geometries can result in nearly pure tones. Examination of the fluid mechanics for the present situation leads to a fluctuating force and thus a dipole source. With the dipole model and the characteristic variables, it is possible to develop an expression for the sound power. The experimental results confirm the relevance of Eq. 4.32.

### 4.10.4 The Vortex Whistle (Rotating Dipole)

When the swirling flow within a pipe encounters the exit, it can become unstable. An example of the original system is shown in Figure 4-30. The instability arises when there is a reversed flow on the axis. The axis of rotation itself precesses about the pipe axis resulting in a rotating force at the pipe exit and a dipole sound field. Studies of this whistle [33, 34] have shown that dynamic similarity based on the characteristic length (pipe diameter) and the characteristic speed (inlet mean flow speed) was not achieved as shown in Figure 4-31. Since the flow precession rate is non-linearly related to the axial flow, a more correct characteristic speed would be that characteristic of the swirl. Carl-Gustav Rossby (1898-1957) developed the ratio of inertial forces to coriolis forces \{A.2.2.5\}. Applied to the present situation it is the ratio of the axial speed to the swirl speed (characterized by a frequency and a length scale). Since the sound frequency is the same as the precession frequency, the Rossby number becomes an inverse Strouhal number. The correct characteristic speed is then \(U=\beta R\), where \(R\) is the tube radius. To test the relevance of this speed to the present situation, the flow rate was increased and the frequency and level of the sound was measured. Figure 4-32 shows the deduced force \{A.2.2.7\} based on the dipole model of Eqs. 4.12. The force should be proportional to \(U^2\) or in this case \((\beta R)^2\). The data scatter was quite small but the speed dependence exponent was between 2.1 and 2.4 suggesting near, but
not quite, dynamic similarity, but certainly close enough to suggest a dipole source. The measurements of the sound field, shown in Figure 4-33, clearly indicate a dipole source.

The model for the vortex whistle sound output is based on the radial dipole of \{4.9.5\}, except that there is only one force and it rotates uniformly, a rotating dipole.

The method of creating the swirl was considered the cause for the lack of dynamic similarity, so the swirl was created with blades inside a straight pipe followed by an expansion within the pipe to create the required axial backflow. Measurements made with this geometry, shown in Figure 4-34, indicate that dynamic similarity was achieved \[35\]. A constant Strouhal number was obtained over a broad range of Reynolds numbers, and for both water and air. This observation resulted in converting a sound study of the vortex whistle to a flow metering device called the swirlmeter. Its accuracy rivals that of the vortex shedding meters of \{4.10.1.2\}, but has a higher pressure drop.

The phenomenon of swirl instability has been shown to occur in other situations \[36\]. One was the flow separation on the upper side of delta-shaped airfoils of high speed aircraft (Concorde) where the slope of the leading edge resulted in a swirl flow that became unstable. A model of this phenomenon is shown in the Figure 4-35. Would there be significant sound generation in this case?

Fig 4-31. The Strouhal number is not constant for the vortex whistle.

Fig. 4-32. The calculated force that creates the vortex whistle sound field.

Fig 4-33. The sound field of the vortex whistle.

Fig. 4-34. Dynamic similarity for the swirlmeter.
Another is the flow within cyclone separators. The swirling flow occurs in an annular region between two tubes. The flow reverses at the closed end of the outer tube and exits through the inner tube. Under certain conditions, the flow in the reversal region becomes unstable, resulting in a period rotating force on the outer tube. In this case, periodic vibration of the unit would indicate vortex instability.

Large centrifugal fans sometimes use radial inlet blades that can be rotated to control the flow into the fan; they create a swirling flow. At near shutoff, where the swirl is very high, *rotating blade stall* occurs. Although not researched, it is highly likely that swirl instability is the cause.

**Key Points:** Swirl instability can occur in several instances and result in dipole sound. The characteristic variables for this phenomenon are more diverse. For the vortex whistle and the swirlmeter, the characteristic length is the tube radius and the characteristic speed is that of the mean flow to which the axial and rotational components are proportional through the swirl angle. For the Concorde (or other delta wing aircraft), the characteristic length is not easy to define; the characteristic speed is aircraft speed.

### 4.10.5 The Edge Tone

There are several sound sources where an alternating vortex flow *impinges* on an object as opposed to forming downstream of an object {4.10.1.1}. Figure 4-36 shows schematically the incompressible circulation of two vortices as they pass the edge. This simple diagram suggests that there is a lateral (lift) force applied to the edge; the result being a dipole flow and...
sound field as shown in Figure 4-37. The diagram also suggests that there is a drag dipole field, identification of which would be a source of information on the relative positions of the impinging vortices.

The source of the vortex motion can be an unstable jet or the wake of an upstream object. The vortex development is alternating, such as shown in Figure 1-8 for an unstable jet. Most sound studies have been of a rectangular jet impinging on linear wedge. In those studies, the feedback from the hydrodynamic sound field (Class I) to the orifice that is then amplified by the unstable jet. A seminal study by Powell of this phenomenon [37] has exposed details of the edge tone phenomenon. A semi-empirical equation for the frequency developed by Curle [38], converted to Strouhal number, is

\[
St_e = \frac{fh}{U} = \left[ \frac{4n+1}{8} - \frac{h}{60d} \right]
\]  

This equation, applicable for \( \frac{h}{d} > 10 \), confirms the choice of the mean speed of the jet at the orifice \( U \) as characteristic speed and the distance \( h \) from orifice to edge (which controls the time for the wave speed of the disturbance to be carried to the edge) as the characteristic length. It also suggests that dynamic similarity is achieved to a first approximation; one deviation is that the correct mean speed (that at the wedge) is less than that at the orifice. The orifice width \( d \) also has some influence; it is related to the vortex size when it impinges on the edge and thus suggests lower frequencies with wider orifices. The integer \( n \) represents the various modes. The presence of a dipole sound field and a periodic force proportional to \( U^2 \) was confirmed [37].

**Key Points:** The edge tone, intrinsic to many sound sources, can be either Class I or II. There are a number of situations where the edge tone geometry is integral to a larger structure adding Class III as another feedback mechanism. Experimental results suggest that dynamic similarity is nearly achieved, but is unlikely to hold for a wide range of variables because the phenomenon is much more complex.

### 4.10.5.1 Pipe Organs

There are several sound sources in which the edge tone phenomenon is associated with a resonant structure that modifies the dipole streamlines. An example is shown in Figure 4-38 where the wedge is part of a tube. A dipole-like flow field is created at the edge (which is an opening in the tube). The vortex flow drives fluid alternately into the tube and then out. The streamlines clearly are distorted from those of the free source. There is a stagnation point opposite the source. The dashed lines, colored in red, are those most strongly modified. The red streamlines in the tube are now augmented by the oscillatory flow in the tube, a superposition of resistive and reactive dipole flow and resistive acoustic flow. The tube length determines whether the tube acoustic pressure or velocity is the dominant influence on the frequency of the tube.

\[ \text{Fig. 4-38. A constrained dipole flow field.} \]
The classic organ is found in churches. The organ pipe is geometrically simple and is driven by an edge tone generator, as shown in Figure 4-39. The plenum in the lower half of the figure supplies the air to produce a jet that impinges on the slit edge. The resulting edge tone couples with the tube (usually a cylinder) in the upper half of the figure. The far end can be either open or closed. Consider the open ended tube. Typical comments on the resonance frequency of such a tube are that \( \lambda = \frac{L}{2} \) for an open-open resonance or that \( \lambda = \frac{L}{4} \) for an open-closed resonance. Simple perusal of the figure suggests that the lower end is neither fully open nor closed, resulting in neither a velocity nor pressure minimum there. Further, it has been shown that the fully open end must have an end correction such as that used by Rayleigh for the Helmholtz resonator: \( \delta = \frac{16R}{3} \). Although most cylindrical pipes are reasonably stiff, mechanical motion has some small influence on the resonant frequency. Hence, organ pipes have been known to have adjustable extensions at the open end.

Here there are two coupled systems, so there are two characteristic scales. For the pipe component, the characteristic length is that of the pipe \( L_1 \), suitably corrected, while the characteristic speed is that of sound \( c_0 \). For the edge tone component, the characteristic length is the orifice to edge distance \( h \) while the characteristic speed is that of the jet \( U \).

It would seem that the maximum oscillatory gain of the system would occur when the preferred pipe frequency matches the preferred edge tone frequency. This relationship is expressed in terms of Strouhal number in Eq. 4.34

\[
St_e = \frac{fL}{c_0}, \quad St_e = \frac{fh}{U}
\]

\[
St_e = St_e \frac{L}{h} \quad M
\]

If dynamic similarity holds for both resonances, the latter equation suggests how organ pipes can be scaled. The apparent simplicity of the equation hides important variable factors such as the effective pipe length \( L_1 = L + a_1 + a_2 \) where \( a_1 \) is correction for the open end and \( a_2 \) is the correction for the end near the jet. The Rayleigh end correction cannot be applicable to either of these conditions. The jet disturbance (vortex) speed from orifice to edge will vary with mean speed \( U \), edge distance \( h \), and slit width \( d \) as suggested in Eq. 4-33. The equation suggests that the jet Mach number and the ratio of effective pipe length to the edge distance are important scaling rules for design of various frequency pipes (to a first approximation).

**Key Points:** The organ pipe is a coupling of two resonant systems so there must be two characteristic lengths and two characteristic speeds. For maximum output, the preferred frequency of each system should be the same so a simple relationship between them can be expressed. However, there are many other factors, such as temperature or jet orientation that

Fig. 4-39. One type of organ pipe.
make the simple formula above only a knowledge framework. The organ is a Class III feedback oscillator.

4.10.5.2 Piccolos, Recorders, Flutes

A number of other musical instruments are based on the edge tone phenomenon. The piccolo, a small version of the flute, is shown Figure 4-40. If blown hard, frequency jumps can occur. The instrument is blown lateral to the tube axis, introducing more flow complexity. The base of the recorder is shown in Figure 4-41. It is blown along the tube axis and is subject to frequency jumps when overblown. Unlike the organ pipe, these instruments have side ports to change the resonance frequency. Essentially, these instruments have been able to compress a number of organ pipes into just one. They are mouth blown, so overtones are under control of the player. The frequencies are determined by Eqs. 4.34, but with a difference. The distance $h$ is constant, but the effective length $L_1$ is determined by the porting. The Mach number is determined by the pressure supplied by the player. Good design of instrument port positions, no doubt, permits the desired frequency to be achieved without excessive changes in the player’s effort. The characteristic lengths conceptually are the same as for the organ pipe, but numerically different. The characteristic speeds are the same as for the organ pipe.

**Key Points:** It is testimony to the skills of early instrument makers that they were able to achieve the right port sizes and positions for a given note without scientific measurement instruments. These instruments are Class III oscillators.

4.10.5.3 Shallow Cavities

Flow over cavities can result in excitation of a feedback loop and narrow band tones. Unlike the edge tone devices noted above, the edge is typically square as shown in Figure 4-42. The flow can connect to various shaped cavities, typically rectangular, and either “shallow” or “deep”. A very large effort has been made over many years to understand and control this phenomenon, since it can occur in the open bomb bays and wheel wells of military aircraft. Shallow cavities have a Class I hydrodynamic feedback mechanism. Figure 4-42 shows an example of the oscillatory vortex flow in such a cavity. One study [39] has shown that several modes of oscillation can occur in a shallow cavity resulting in an empirical equation, now called Rossiter’s formula. Lee and others [40] have shown it in Strouhal number form as Eq. 4-35.
The characteristic length is cavity length $h$ (not depth) and the characteristic speed is that of the free stream $U$. The bracketed term includes two loop speeds; the downstream speed is the speed of the vortices $U_v$, and the upstream speed is that of sound. The various modes are described by an integer $n$ with an empirical delay constant $\beta$ (near 0.25). The integer is closely related to the number of vortices enroute to the edge.

An example of frequency measurements is shown in Figure 4-43. The variable $L$ is the same as the $h$ in Eq. 4-35. It is interesting to note that for the first mode, the Strouhal number is not much higher than that related to vortex shedding from a cylinder \{4.10.1.1\}. Examples of the sound field for several modes are shown in Figure 4-44. It is clear that the fluctuating force at the downstream edge is the source. Since the Mach number of the flow can be appreciable, refraction makes it difficult to determine the major axis of the dipole-like sound field. However, a simple minded approach is to consider that a vertical force is exerted on the upper surface and a horizontal force is exerted on the vertical surface immediately inside the cavity. If these forces are of similar magnitude and reasonably in phase, one would expect the dipole axis to be inclined at forty five degrees in the upstream direction. Not surprisingly, the angle is near that value, but the apparent source is located downstream of the edge. The presence of cavity flow strongly modifies the frequency that would be preferred by the free edge tone.

**Key Points:** Extensive research has defined the proper characteristic length and speed, despite the fact that dynamic similarity is not achieved in such a complex flow phenomenon. The Strouhal number is a weak function of the Mach number (Reynolds number actually). Unlike the organ pipe, there is no distinct acoustical resonance of the cavity. It is a Class I hydrodynamic feedback system.

\[
St_n = \frac{f_n h}{U} = \frac{n - \beta}{U \left( \frac{1}{c_0} + \frac{1}{U_v} \right)}
\]
4.10.5.4 Deep Cavities

An exterior flow or flow in a duct with a “deep” side branch can excite resonance in the branch. Often the branch is intended to be a narrow band filter, and design is often based solely on acoustical concepts. The presence of flow can change the side branch from a muffler to a sound source. This situation is very similar to that of the organ pipe. There are two preferred frequencies; that associated with the edge tone and that associated with the cavity. Selamet and others [41-43] have made extensive studies of this phenomenon in a duct and its application to engine intake sound. There are two sets of characteristic variables. For the edge tone, the characteristic length is the side branch width \( h \) and the characteristic speed is that of the flow in the duct \( U \). For the side branch, the characteristic length is the side branch depth \( L \) and the characteristic speed is that of sound \( c_0 \). Unlike the organ pipe, one end is completely closed. The open end can either radiate into a free space or into a duct. In the former case, the Rayleigh end correction may be used as a first approximation to correct for the monopole-like radiation. In the latter case, an unknown correction must be applied. For the present purposes, an arbitrary constant \( \beta \) will be used to represent the correction. The side branch Strouhal number then can be expressed as

\[
St_c = \frac{fL_c}{c_0} = \frac{(2m+1)}{4} \tag{4-36}
\]

\[
L_c = L \left( 1 + \beta \frac{h}{L} \right)
\]

The letter \( m \) is an integer and \( \beta \) is an end correction. Again, dynamic similarity is achieved to a first approximation. Eq. 4-34 can be used to relate the two Strouhal numbers, yielding an expression for the relationship between the three important variables for (presumed) maximum output.

\[
\frac{L_c M}{h} = \frac{4m + 2}{4n + 1} \tag{4-37}
\]

**Key Points:** Again, dynamic similarity is achieved to a first approximation when the characteristic variables for the cavity (side branch) are properly defined. The relationship between the three important variables is seldom an integer. Deep cavities have a Class II acoustical feedback mechanism.

4.10.5.5 Bottles

Blowing over the edge of a bottle can create a nearly pure edge tone of low frequency. The geometry is similar to, but not the same as, each of the previous edge tone/cavity situations. Are the preferred edge tone frequencies coupled to the preferred frequencies of a longitudinal cavity? Another way to frame the question is to ask if a Helmholtz resonator is just a variation of the previous devices. A reasonably complete theory of Helmholtz resonators [44] took into account the reactive and radiative exterior end corrections as well as the evanescent reactive higher modes in the interior of the cavity. The resonance equation is shown in Eqs. 4-38. It is a transcendental equation where \( A_c \) is the cross sectional area of a cylindrical cavity of depth \( L \). \( A_o \) is the area of the orifice of depth \( L_o \). \( \delta_e \) is the exterior end correction, \( \delta_i \) is the interior end correction, and \( kL \) is the Helmholtz number.
Given the areas, the characteristic length is the cavity depth \( L \), and the characteristic speed is that of sound as with the cavity. A simple test was performed on a two liter (122 in\(^3\)) bottle with a 1 3/8 inch diameter opening. The frequency was close to 140 Hz. The cavity was close to 9 inches deep giving a one-quarter wavelength frequency near 375 Hz. It is clear that the Helmholtz resonance is not a longitudinal resonance, despite the similar characteristic variables. The Helmholtz number was near 0.58 (\( St=0.09 \)), suggesting that the edge tone was not the controlling motion. The difference, of course, is the presence of a neck. If the area of the orifice equals that of the cavity, the equation degenerates to one describing a straight pipe with the addition of internal cross modes not usually accounted for in pipe resonance calculations.

**Key Point:** Despite the fact that the characteristic variables are the same as for a straight pipe, and the excitation was fluid mechanical, the geometric changes create a markedly different result.

### 4.10.5.6 The Police Whistle

There are a number of devices used by police and others to create piercing sounds. The London Metropolitan police use a linear whistle, more like a small recorder. More common, however, is the whistle shown in Figure 4.45. In this device, the cavity is a closed end cylinder (3/4 inch diameter), but with the cylinder axis lateral to the jet. The orifice on whistles of this type is 3/4 by 1/16 inch and spaced 1/4 inch from the edge. When blown weakly, the sound is narrow band random. When blown more forcefully, the tone is established in the 2500 Hz one-third octave band, quite pure and quite loud; adjacent bands are at least 20 dB down. The level of the tone increases with how hard it is blown, and the frequency increases only slightly suggesting hydrodynamic feedback. Considering the edge tone geometry noted in \{4.10.5.2\}, one would expect several jumps in frequency, but none occur. Lovers of dynamic similarity hate this device.

Although complete understanding of whistle operation is not the central issue here, the author is not aware of any scientific research on this whistle. It seems clear that the rotating flow in the cavity (interior vortex) is the controlling influence. When the jet moves toward the cavity an additional thrust is given to the interior flow, which then rotates around and back to the edge, forcing the jet to move away from the cavity. The reduced flow rotates around, allowing the jet to return. The fluctuating force on the edge creates the dipole-like sound field.

The central issue is the characteristic variables. It is highly likely that it is a purely Class I whistle controlled by hydrodynamic feedback. Since there is a boundary layer in the cavity, the maximum angular speed of rotation is at a lesser radius than that of the cavity. The characteristic variables in the cavity are the radius \( r \) and the rotation speed \( \omega \). The characteristic variables for the jet are the gap distance \( h \) and the mean jet speed \( \bar{U} \). It is likely that the boundary layer speed

\[
cot(kL) = \frac{A_e (L + \delta_e + \delta_i)}{A_o} k L
\]

\[
kL = \frac{fL}{c_o} = 2\pi St
\]

Fig. 4-45. The police whistle.
is restricted by viscous effects, so that the cavity characteristic speed is a weak function of Reynolds number.

**Key Points:** The police whistle is unique among edge tone devices in that there are no jumps when the tone is established. The characteristic variables seem well defined, but dynamic analysis fails and so understanding must await a more detailed study.

### 4.10.5.7 The Levavasseur Whistle

The cross-section geometry of one version is shown in Figure 4-46; it is an axisymmetric version of the police whistle that has been modified to have two cavities. Another variation of this whistle is to add a toroidal horn to better couple the output to the environment. It has been reported that the sound is very intense. This result is not surprising, since the simple police whistle has high output. To the author’s knowledge, no scientific study has been done to elucidate the detailed mechanisms of its operation. The jet speed can be supersonic for especially high levels. It is highly likely that the two cavities are in anti-phase.

**Key Point:** Although this whistle is a Class I hydrodynamic feedback edge tone device, its detailed operation, like the police whistle, is still an enigma.

### 4.10.6 Rectangular Supersonic Jets (Screech Tone)

When a rectangular jet emerging into ambient medium has a pressure ratio greater than the critical, the flow becomes supersonic on exit. For an air jet emerging into ambient air, the ratio is 1.893 (about 27.8 psi). This phenomenon can occur in engine exhausts, pressure relief valves and jet engines. The flow expands in an attempt to adjust to the new environment with an expansion wave. The reflection of internal expansion wavelets cause it to contract into a shock, in an attempt to return into the original subsonic exit speed. This forms what is called a shock cell. The process is repetitive, creating a series of shock cells and one might describe it as a spatial oscillation. These can be seen in the exhaust of rockets. As subsonic jet flows can be unstable, supersonic flows are also unstable. In rectangular jets, the instability shows as asymmetric cell distortions, similar to subsonic jets. The periodic sound from these jets is called a screech tone and is extremely powerful. Powell [45] first described the
phenomenon and because of application to military aircraft and potential structural fatigue, much subsequent work has been done. The process is similar to edge tones in that a disturbance at the orifice is amplified and carried downstream resulting in lateral momentum fluctuations that apply a net force to the surrounding medium and resulting in a dipole sound field. By analogy, one might say that the shock cells are the edges over which the momentum fluctuations occur. The sound field returns to the orifice to institute further disturbances. The sound energy is sufficient for the field to show in a shadowgraph; Figure 4-47 shows a shadowgraph photo (by M.G. Davies) for a rectangular supersonic jet in sinuous mode. The photo clearly shows the sound field and the phase reversal on either side of the jet. In the sinuous (asymmetric) mode there is lateral motion of the shock cells. A symmetric mode has also been found. Supersonic flows can be quite complex and some tentative explanations are available [45, 46, 47]. As with hole and ring tones [3.12.1, 4.10.3], these jets are sensitive to local sound reflecting surfaces.

**Key Points:** Exhaust flows above the critical pressure ratio can result in intense single frequency sound. Depending on the shape of the orifice, the source can vary from a dipole to another descriptor [5.3.2]. The characteristic speed is that in the exit plane, to which the variable velocities in the shock cells are proportional. The characteristic length is usually the nozzle diameter, to which the cell dimensions are proportional. The asymmetry of the sound field shown in Figure 4-47 makes it a dipole source with its axis lateral to the jet flow. Aside from combustion [3.11.1], this is the first instance where sound is generated without the fluid interaction with a solid, a truly aerodynamic source.

### 4.10.7 Circular Supersonic Jets (Screech Tone)

A similar phenomena has been found to occur with circular supersonic jets [48]. In this case there can be three modes of motion: symmetric (torroidal), asymmetric (sinuous), and helical. Although the presence of such a loud single frequency needs to be noted here, no further analysis of the characteristic lengths and speeds are provided at this time.
Chapter 5
Quadrupole Sources

5.1 The Mathematical Model

The basic wave equation for the quadrupole includes dependence on all three directions

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \psi^2} - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} (r \phi) = 0
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \psi^2} + k^2 \Phi = 0
\]

(5.1)

\(\phi(r,\theta,\psi,t)\) and \(\Phi(r,\theta,\psi,\omega)\) are the velocity potentials. Both polar angles need to be taken into account for this case. The key physical variables are

\[
p(r,\theta,\psi,t) = \rho_0 \frac{\partial \phi}{\partial t} \quad u_\phi (r,\theta,\psi,t) = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}
\]

\[
u_r (r,\theta,\psi,t) = -\frac{\partial \phi}{\partial r} \quad u_\psi (r,\theta,\psi,t) = -\frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \psi}
\]

(5.2)

5.1.1 The Quadrupole as a Merging of Two Dipoles

The monopole in Chapter 3 had no preferred direction (the source was scalar). The dipole in Chapter 4 was the joining of two monopoles of opposite sign oriented along the z axis. The source was a force vector in that direction. The next step is the joining of two dipoles of opposite sign. Since we are dealing with two vectors, they can be oriented arbitrarily with respect to each other. When they are, the source is called an oblique quadrupole. Such a source can be resolved into two perpendicular components. One is called a lateral quadrupole where the orientation of one axis is perpendicular to the other. The second is called a longitudinal quadrupole where both axes are parallel \{1.3\}. In fluids, the lateral quadrupole is associated with shear flows and most studies suggest that it is this form that predominates in the sound generated.

5.2 The Single Frequency Point Lateral Quadrupole

5.2.1 Physical Variables

For this case, one axis is perpendicular to the other. The velocity potential is

\[
\phi_{lat} = \frac{Q_{lat}}{4\pi} \left[ \frac{3(1 + ikr) - k^2 r^2}{r^3} \right] e^{i(\omega t - kr)} \cos \theta \cos \psi
\]

(5.3)
Knowing that the dimensions of the velocity potential must be $L^2/T$, the dimensions of the quadrupole source strength must be $L^5/T$. The physical interpretation can be clarified best by first developing the two significant physical variables. Using the basic equations for these variables $\{A.3.2\}$, they are

$$p = \frac{ikZ_0Q_{lat}}{4\pi} \left[ \frac{3(1+ikr)-k^2r^2}{r^3} \right] e^{i(\omega t-kr)} \cos \theta \cos \psi$$

$$u_r = \frac{Q_{lat}}{4\pi} \left[ \frac{9(1+ikr)-4k^2r^2-ik^3r^3}{r^4} \right] e^{i(\omega t-kr)} \cos \theta \cos \psi$$

(5.4)

The other velocity components can be determined by carrying out the differentiations indicated in Eqs. 5.2.

5.2.2 Near Field

With $kr \ll 1$, both sound pressure and radial velocity increase with high powers of radius. This suggests that quadrupole action is much more “near field” than the lower order sources as shown in the equations to the right.

$$p = \frac{i3kZ_0Q_{lat}}{4\pi r^3} e^{i(\omega t-kr)} \cos \theta \cos \psi$$

$$u_r = \frac{9Q_{lat}}{4\pi r^4} e^{i(\omega t-kr)} \cos \theta \cos \psi$$

5.2.3 Far Field

The sound pressure and radial velocity decrease inversely with increasing radius. When $kr \gg 1$, the equations reduce to those shown on the right.

$$p = \frac{-ik^3Z_0Q_{lat}}{4\pi r} e^{i(\omega t-kr)} \cos \theta \cos \psi$$

$$u_r = \frac{-ik^3Q_{lat}}{4\pi r} e^{i(\omega t-kr)} \cos \theta \cos \psi$$

5.2.4 Radiation Impedance

The expression for radial acoustic impedance is complicated for arbitrary distances from the source; it is given in Eq. 5.5 and is shown graphically in Figure 5-1. The figure clearly shows how large the reactive impedance is with respect to the sound (resistive) part at close distances. Quadrupole impedance is compared with the impedance of the lower order sources in $\{6.2\}$.

$$Z_r = \frac{k^6r^6+i\left(27kr+6k^3r^3+k^5r^5\right)}{81+9k^2r^2-2k^4r^4+k^6r^6}Z_0$$

(5.5)

Fig. 5-1 The radial impedance of a lateral quadrupole.
5.2.5 Sound Intensity

The radial intensity is

\[ I_r(r, \theta, \psi) = \frac{Q_{\text{lat}}^2 Z_0}{16\pi^2} \left( \frac{k^6 r^6 + ikr \left( 27 + 6k^2 r^2 + k^4 r^4 \right)}{r^8} \right) \cos^2 \theta \cos^2 \psi \]

(5.6)

\[ I_r(r, \theta, \psi) = \frac{k^6 Q_{\text{lat}}^2 Z_0}{16\pi^2 r^2} \cos^2 \theta \cos^2 \psi \]

The second equation is the far sound field approximation for \( kr >> 1 \). The resistive component of the radial intensity increases with a high power of the frequency. The reactive component of the radial intensity is considerably larger at low frequencies suggesting more incompressible flow components than the dipole or monopole source. The other components of intensity are totally reactive creating a fluid loading of the source; they have a complex dependence on angle.

The change in the resistive and reactive components (the bracketed term in Eqs. 5.6) of the radial intensity with distance and frequency is shown in Figures 5-2 and 5-3.

The resistive impedance decays with the square of distance while the reactive impedance decays rapidly and then with the cube of distance. Note that at low frequencies and close-in distances, the sound field is buried in the reactive (hydrodynamic) motion. The directivity of the source is considerably more complicated than for the lower order sources. This latter comment is based on one source that is fixed in space. In fluid flows neither of these conditions apply, so directivity information is not helpful {5.5.1}. If the source does meet the conditions, the complexity of the sound field can easily separate it from dipole or monopoles.

**Key Point:** Point lateral quadrupoles are inefficient sources of sound compared with the lower order sources and have more complex sound field directivity.
5.2.6 Estimates of Sound Intensity

The estimated sound intensity is given in the first of Eqs. 5.7. The ratio of the sound intensity estimate to the actual (real) part of the radial intensity is given in the second equation.

A graph of the ratio is shown in Figure 5-4. The error exceeds 1 dB when kr is less than 4. This is a greater error restriction than for the dipole source. The error can be quite significant when close to the source. For example the minimum distance is 137 inches at 63 Hz, and 34 inches at 250 Hz.

Since the range of the near field is greater at low frequencies, spectrum distortion occurs if the measurement is made too close to the source.

\[
EST[I_r] = \frac{p \cdot p^*}{Z_0} = \frac{Z_0 Q_{lat}^2}{16\pi^2} \left[ \frac{9k^2 r^2 + 3k^4 r^4 + k^6 r^6}{r^8} \right] \sin^2 \theta \cos^2 \psi \quad \text{Ratio} = \frac{9 + 3k^2 r^2 + k^4 r^4}{k^4 r^4}
\] (5.7)

Fig. 5-4. The error in estimating lateral quadrupole sound intensity from sound pressure measurements.

5.2.7 Sound Power

Integrating the far field radial intensity over the surrounding volume yields Eq. 5.8

\[
W_{lat} = \frac{k^6 Q_{lat}^2 Z_0}{24\pi}
\] (5.8)

This relationship implies that the sound power increases with the sixth power of the frequency forever! Aside from the restriction that the source is a point, does this dependence make sense? The quantity \( Q_{lat} \) is convenient mathematically, but has little physical significance. This needs to be corrected.

5.2.8 Source Strength Interpretation

Look for any monopole component (volumetric flow rate). The result of integrating the radial velocity is

\[
Q_{lat} = \int \int u_r r^2 \sin \theta d\theta d\psi = 0
\]

The \( \psi \) integral is zero so \( Q_{lat} \) is not a volumetric flow rate as might be surmised from its dimensions \( L^5/T \).
Look for any dipole component (net force acting on the medium). The differential forces are

\[ dF_z = pr^2 \cos \theta \sin \theta d\theta d\psi \]
\[ dF_x = pr^2 \sin^2 \theta \cos \psi d\theta d\psi \]
\[ dF_y = pr^2 \sin^2 \theta \sin \psi d\theta d\psi \]

When the angular dependence of the sound pressure is added to these terms and the integration performed, each integral equals zero. There is no net force acting the medium.

For the monopole, the source was scalar (no preferred direction), and for the dipole, the source vector (one preferred direction). The quadrupole is another order higher (two preferred directions), a second order tensor. Applied stress has this property. Figure 1-5 in Chapter 1 shows that two opposing forces displaced laterally make a shear moment resulting in a lateral quadrupole. The geometry of that figure suggests that one preferred direction \( z \) is in the direction of the opposing forces and the second direction \( x \) is perpendicular to it. For that the tensor symbol \( \tau_{xz} \) will be used. To calculate the moment created by the forces, convert the \( z \) coordinate to polar coordinates, use the differential forces given above and perform a spatial integration of the moment arm close to the source. When this is carried out, the source \( Q_{\text{lat}} \) \((L^2/T)\) is replaced by \( \tau_{xz} \) \((FL)\) and the far sound field equations become

\[ I_\nu (r, \theta, \psi) = C_1 \frac{k^4 \tau_{xz}^2}{Z_0 r^2} \cos^2 \theta \cos^2 \psi \]
\[ W_{\text{lat}} = C_2 \frac{k^4 \tau_{xz}^2}{Z_0} \]

The \( C \) values are constants and are not pertinent to the development. By analogy to Eq. 3.5 for the monopole and Eq. 4.7 for the dipole, the sound power of the lateral quadrupole can be expressed in the form below.

\[ W_{\text{lat}} = \frac{C_3}{Z_0 c_0} \left( \frac{\partial \tau_{xz}}{\partial t} \right)^2 \]

*The sound from a lateral quadrupole is created by the mean square of the time rate of change of the shear stress rate.*

### 5.2.9 Dimensional Analysis

By introducing dimensionless factors into the sound power equation, it becomes

\[ W_{\text{lat}} = \frac{K \rho_0}{c_0^5} \tau_{xz}^2 S^4 U^8 L^2 \]

\[ \hat{W}_{\text{lat}} = K \tau_{xz}^2 S^4 M^5 \]

See \{A.2.2\} for definitions of the dimensionless ratios. This equation applies for point-like lateral quadrupoles. This is the well known \( U^8 \) law for jet noise. Because of the high
exponent on the Strouhal Number, if it depends on Reynolds number, the $U^8$ speed law may not be achieved, \{6.4.2\}. A similar caution applies to the shear moment. 

**Key Points:** The dependence of the point lateral quadrupole on frequency and speed is considerably higher than for the monopole or dipole, suggesting this source is important at higher frequencies and speeds. No derivation for the finite lateral quadrupole is given, since there is little evidence that finite size effects are important in the major application of this source: jet noise.

### 5.3 The Single Frequency Point Longitudinal Quadrupole

The development of the longitudinal quadrupole is unnecessary for this monograph since it appears that it is not an important factor in high speed flows. The primary features of this quadrupole is that it has a more complex near field, and the intensity has a $\cos^2 \theta$ directivity. The form for the sound power (Eqs. 5-9, 5.11) is the same as that for the lateral quadrupole. However, the constant is a factor of three greater than $C_2$ for the lateral quadrupole.

### 5.4 Modeling Quadrupoles

The basic feature of the theoretical quadrupole is a highly directional sound field resulting from stresses being applied to the surrounding fluid. Some real sources may meet these requirements but others do not, so may be described only as *quadrupole-like*.

1. *Fluctuating stresses in free space.* They create a multi-directional sound field.
2. *Fluctuating stresses along a surface.* The orientation of the stresses determine their interaction with the reflected image source.

The stresses are related to fluctuations in the local momentum flux. Generally, the only significant quadrupole sources are those generated by turbulent fluid flows. The size of turbulent eddies appears to be quite small relative to the wavelength of the sound generated by them, so the sound power equation developed based on a point source (Eq. 5.11) has been verified experimentally \{12, 13\}. The difficulty in applying quadrupole concepts to determine sound field directivity is that the turbulent eddy structure is randomly oriented and embedded in a high speed flow with large velocity gradients that cause refraction.

### 5.5 Modeling Category I Quadrupole Sources

#### 5.5.1 Subsonic Jets

The theoretical models have each of their axes in a defined direction, so one would expect to be able to measure source directivity. Unfortunately, not many higher order sources have such fixed directivity. The initial interest in these sources was in the noise from jet engines that was determined to be of quadrupole nature. The flow from a subsonic jet is highly turbulent, so the source orientations are more nearly random; one cannot expect to derive any information about the source from directivity measurements. The theoretical models are in a stationary medium which is decidedly not the case for jet exhausts from high speed aircraft. Further, the jet structure changes with distance from the exhaust plane, so the characteristic scales change with space. Despite these severe limitations, it is possible to learn some things.
about the sound from a high speed flow exhausting from a nozzle. This is another example in which the characteristic scales are a function of position.

Figure 5-5 shows a shadowgraph of a turbulent jet clearly showing the radiated sound field. The figure shows that the major sound sources are not far from the nozzle exit, but gives little information about the frequency spectrum or speed dependence. To learn more, consider the flow from a circular nozzle; it has three regions as shown in Figure 5-6.

### 5.5.1.1 Core Region

The central area of the exit flow may or may not be turbulent, but most importantly the boundary layer will be highly turbulent. Upon exit, that boundary layer grows until the central area consumed. The outer edge of the boundary layer slowly increases radially while the inner edge radius decreases until the central region disappears in about four nozzle diameters. This disappearance is the end of the core region. The principle characteristic of this region is that the velocity profile of the mean flow has a flat central contour equal to the exit velocity. For this region, the characteristic length is the thickness of the boundary layer which is an almost linear function of distance from the exit. The characteristic speed is the speed of the central core.

![Fig. 5-5. A shadowgraph of jet flow showing the sound field.](image)

Consider that the turbulent region is composed of several annular sections, each composed of quadrupole sources. Since the volume of the annular region grows with downstream direction, the size of turbulent eddies must grow and the frequency characteristic of them must lessen. Consider that the core region exists for four diameters then the following approximate relationships on the right apply. The initial boundary layer \( \delta_0 \) is that formed within the exhaust pipe whose diameter is \( D \). The variable \( x \) runs from zero to \( 4D \). The cross sectional area of the radiating turbulence is an annular region.

![Fig. 5-6. The velocity profiles of a high speed jet.](image)
that is small at initiation and encompasses the entire jet at the end of the core region. A local volume is defined as the local area times an increment of \( x \). The radiation is from turbulent eddies of a specific size so that the volume is occupied by a finite number of quadrupole radiators. Eddies increase in size directly with the local volume so they occupy a larger fraction of total local volume as the boundary layer grows. The eddies are presumed to be uncorrelated in each local volume so their powers are mean square additive. Eqs. 5-11 is used to calculate the sound power of each local volume (annular section) and the volumes are summed as independent radiators.

An example calculation was done for a nozzle diameter of two feet and exit velocity of 500 ft/sec. Twenty local volumes were summed in the core region to provide relative overall levels. The results, shown in Figure 5-7, are of arbitrary level intended to show the relative contributions to output along the core region. The initial level is the lowest since the annular radiating volume is extremely small relative to downstream volumes. The level rapidly approaches a constant level with distance, suggesting that the entire core region participates significantly in the sound output.

If similarity holds, the Strouhal number is relatively constant over the core distance. The characteristic length is that of the growing boundary layer and the characteristic speed is approximately that of the central core. The maximum frequency of each local volume decreases with distance along the core. The relative spectra of each of the twenty volumes were summed and the resultant one-third octave band spectrum is shown in Figure 5-8. The spectrum contour was that of a haystack spectrum with 6 dB/octave slopes on each side of the maximum. Close to the orifice, the output has higher frequencies, but a lower level. The output further downstream has lower frequencies but a higher level so it masks the high frequency contribution. Surprisingly, for such a simple use of dynamic similarity, measurements of aircraft passes, show spectra that have a maximum near that shown in the figure with similar spectrum contours.

**Key Point:** The core region is a large contributor to sound power. This is supported by the shadow graph of Figure 5-5.
5.5.1.2 Transition Region

The velocity profile at the end of the core region is similar to that shown in Figure 5-6. The curvature of the profile changes in the transition region. It is more difficult to analyze this region, except to know it must patch together the two regions around it. The characteristic length and speed are difficult to define here.

5.5.1.3 Fully Developed Region

The velocity profile for fully developed jet flows is similar to that shown in Figure 5-6. The width of the jet increases, and the centerline velocity decreases with distance from the core region. This region is the only example in this monograph where both of the characteristic scales vary both with downstream distance and lateral distance. Data from measurements suggest a lateral profile similar to the \( U_c \sec h(w) \) function where \( w \) is the width (a function of axial distance), and the decay of \( U_c \) is linear with distance. To avoid getting into mathematics more appropriate to research, the lateral profile was subdivided into nine annular regions, much like a multi-tiered cake. The nine annular regions were calculated for positions from \( 5<x/D<8 \) with increments of \( x/d=0.4 \). Essentially, the model was a multi-dimensional summation. Again, without data on eddy intensities or sizes, the scaling rules were used to create the figures below.

Figure 5-9 shows the trend of sound power laterally at \( x/D=5 \). The transition region was set between \( 4<x/D<5 \). It is clear that since characteristic speed decays rapidly with lateral distance, sound production also falls off significantly. This estimate is based on similar eddy intensities at every lateral position, which is not likely to be correct, since shear on the axis is less. However, the turbulence in the central area along the axis is likely to be the major source of sound at any axial position.

Figure 5-10 shows the trend of sound power in the axial direction, starting at five diameters downstream just at the end of the transition region. The decay is about 18 dB/doubling of distance. The central area grows with distance, as does eddy size, but the speed dependence determines the sound power.

**Key Points:** This section was not intended to provide a detailed analysis of the sound generated by a subsonic circular jet; that has been the subject of much research. Were any of the magnitudes accurate? No. That kind of information must be determined by detailed theory and experiment. It might be noted that several ad hoc approximations as to eddy size and boundary layer growth were needed, but knowledge of the local characteristic
scales and the quadrupole power law overcomes these limitations. Considerations of directivity were omitted. Not only are the stresses randomly oriented, but velocities are sufficiently high that the sound is redirected by velocity and temperature gradients.

If the thrust is kept constant, the exit velocity is halved when the diameter of the jet is doubled. One-half velocity implies a 24 dB reduction in sound emission! Early noise reduction methods attempted to change the flow structure by corrugating the outer edge of the exhaust nozzle to mix the exterior flow with the engine flow. Unfortunately, drag was increased and performance was reduced. Current methods use high-bypass turbofans with greater thrust, fuel efficiency, reduced exit speed and reduced sound generation.

**Key Points:** The purpose was to show how a knowledge framework can be built around the sound radiated by jets with very few tools. In particular, it is known that the sound of high speed jets is quadrupole in nature, and the overall velocity profiles are known. With these two pieces of information, along with scaling rules, it was possible to develop a simple, but reasonable, framework for where the sound was most likely to be emitted. Unlike earlier chapters, this was accomplished by letting the characteristic variables be *functions of space*.

### 5.5.2 Supersonic Circular Jets (Screech Tone)

Because circular jets can radiate in several modes, this section supplements that in {4.10.7}. When a jet emerging into ambient medium has a pressure ratio greater than the critical, the flow becomes supersonic on exit. Details of the concept are given in {4.10.6} and there the rectangular jet shock cells had asymmetric distortions. Here the jet is circular and the instability results in either symmetric or helical distortions of the shock cells. The symmetric distortion occurs at Mach numbers less than 1.25 and the apparent source is near five cells downstream. The helical distortion occurs at greater Mach numbers and occurs near three cells downstream (Are we dealing with a version of the vortex whistle? {4.10.4}). Considering an analogy to the basic source types, in the rectangular jet the center of mass of the cells oscillate laterally giving rise to a dipole-like sound field. In the circular jet with symmetric distortions the volume of the cells vary giving rise to a monopole-like sound field. In the circular jet with helical distortions, the sound field is more like that of a rotating dipole source. A simple equation for the Strouhal number of the screech is

\[
St = \frac{fL}{c_0} = \frac{0.7}{u + c_0}
\]

The characteristic length L is the cell length, and the characteristic speed is that of sound. Considering that both speeds are almost equal, a Strouhal number of 0.35 results. Some experimental data have indicated an almost identical value for a Strouhal number based on the orifice diameter as the characteristic length and jet exit speed as the characteristic speed, suggesting the close relationship between the variables.

**Key Points:** Assigning characteristic variables can be difficult. The flow leaving the nozzle will be sonic, expanding to supersonic in the area critical for sound generation, but since one is dependent on the other, the nozzle speed is a reasonable characteristic speed. Since it is eddies passing through the shock cells that cause the disturbance, it would seem that the characteristic length would be cell spacing. As with all free jet flows, the characteristic variables are a function of position in the jet.
Chapter 6
Comparing Sources

6.1 Four Limiting Distances

6.1.1 Validity of the Wave Equation

Any calculations or measurements must be done beyond the hypothetical surface that separates the volume in which the wave equation is valid and the one in which it is not. Approximation 9 \{D.4\} is the first one that invalidates the wave equation as the source is approached. The numbers shown in Table 6-1 are the radii at which the condition shown in the equation on the right is met. Note that the amplitude of the source plays an important role and since that is probably not known, the rule is simple: stay as far away as possible being aware that erroneous results can occur if the criterion is not met.

<table>
<thead>
<tr>
<th>Source</th>
<th>Monopole</th>
<th>Dipole</th>
<th>Quadrupole</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius outside of which Wave equation is valid</td>
<td>$r_c = \frac{3Q_m}{2\pi\omega}$</td>
<td>$r_c = \frac{4F_z}{\sqrt{2\pi\rho_0\omega^2}}$</td>
<td>$r_c = \frac{3\sqrt{96\pi\omega}}{\pi^2\rho_0\omega^2}$</td>
</tr>
</tbody>
</table>

*Table 6-1. The radius at which the wave equation becomes invalid.*

6.1.2 The Far Sound Field

Within a certain radius estimates of sound intensity from sound pressure measurements are not accurate. The radius at which the error is 1 dB is shown in the Table 6-2 below.

<table>
<thead>
<tr>
<th>Source</th>
<th>Monopole</th>
<th>Dipole</th>
<th>Quadrupole</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius for 1 dB error.</td>
<td>No Limit</td>
<td>$r_c = \frac{2c_0}{\omega}$</td>
<td>$r_c = \frac{4c_0}{\omega}$</td>
</tr>
</tbody>
</table>

*Table 6-2. The radius at which sound pressure measurements diverge from sound intensity.*

6.1.3 The Far Geometric Field

Eq. 3.18 (Chapter 3) shows the distances for each of two point monopoles near each other. The approximation displayed is used for phase, but the two distances are presumed to be equal. The ratio of $r_1$ to $r_2$ is assumed to be 1. Comparison of the correct amplitudes with 1 shows that to have an error less than 1 dB for mean square pressure, the radius must be ten times the spacing between the sources as shown on the right.

$$\frac{r}{2h} \geq 10$$
6.1.4 Finite Source Size

It is convenient to consider the sources to be sufficiently small with respect to a wavelength, that they can be considered point sources. For example, the dynamic similarity equations are based on this assumption (Eqs. 6-1). Where does this approximation fail? The frequency above which the error is greater than 1 dB is shown in Table 6-3 below.

<table>
<thead>
<tr>
<th>Source</th>
<th>Monopole</th>
<th>Dipole</th>
<th>Quadrupole</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency for 1 dB error.</td>
<td>$f \geq \frac{c_0}{4\pi a}$</td>
<td>$f \geq \frac{c_0}{4\pi a}$</td>
<td>$f \geq \frac{c_0}{4\pi a}$</td>
</tr>
</tbody>
</table>

*Table 6-3. The frequency above which the finiteness of the source is important.*

6.2 Radiation Impedance

Each of the sources below a value of $ka=1$ has a variable dependence on frequency. Figure 6-1 shows this dependence as a fraction of the characteristic impedance. The monopole impedance rises with $f^2$, the dipole with $f^4$ and the lateral quadrupole with $f^6$. At low frequencies, the monopole is orders of magnitude more effective that the higher order sources, hence one would expect that monopoles would dominate at low frequencies.

![Fig. 6-1. The resistive radial radiation impedance for each of the three basic sources.](image)
6.3 Speed Dependence

Each of the sources has a different dependence on speed. The dimensional and dimensionless sound power equations for each (point) source type are given in Eqs. 6.1.

$$W_m = A_m \dot{Q}^2 S^2 U^4 L^2 \quad \hat{W}_m = C_m \dot{Q}^2 S^2 M$$
$$W_d = A_d \hat{F}^2 S^2 U^8 L^2 \quad \hat{W}_d = C_d \hat{F}^2 S^2 M^3 \quad (6.1)$$
$$W_q = A_q \tau^2 S^4 U^8 L^2 \quad \hat{W}_q = C_q \tau^2 S^4 M^5$$

In most practical situations, the source size is small with respect to the wavelength of the sound emitted so use of the point source model is relevant. Since each of the dimensionless source strengths will be different, an example of what might happen for a specific application (exhaust flow from a tube), as speed is increased, is shown in Figure 6-2. At low speeds, such as that from an automobile, the monopole component (oscillatory mass flow) dominates. As the speed increases the dipole component (turbulent boundary layer flow) begins to dominate. This can be observed with motor homes under high acceleration. At very high speeds, the quadrupole component (turbulent jet flow) dominates.

![Fig. 6-2. The major sound source may change with speed.](image-url)
6.4 Deviations from the Speed Laws

One powerful method of identifying the dominant source type is to vary speed. Unfortunately, there are several reasons for the measured speed dependence to vary from that of the theoretical models. Nature, liking only ratios, (Strouhal number) and measurements liking specific frequencies, means fixed bandwidths can be a cause of deviations. The similarity equations omit the influence of viscosity (Reynolds number). The dimensionless magnitudes (volumetric flow rate, force, or stress) very likely have dependence on Re, particularly in turbulent flows where viscosity influences the scale of turbulence. The same can be said of the Strouhal spectrum. Near field measurements can influence results; pressures at low frequencies overestimate the actual level.

Detailed analysis or experiment is required to define this deviation, so only simple examples of the consequences are given below.

6.4.1 Measurement Bandwidth

Measurements are tied to specific numbers, such as frequency, while nature is tied to ratios of forces, such as the Strouhal number, which is generally constant. An example of this type of deviation is the speed dependence of a typical large air handling fan. There is sufficient evidence to conclude that fan sound is dominated by dipole sources at the fan rotor. One would expect a $U^6$ speed dependence, but measurements do not confirm it. Figure 6-3 on the right shows how a spectrum changes level and frequency as speed changes. The width of the box represents the fixed frequency range of measurements; the height represents level. The actual spectrum of the sound is shown as a horizontal arc; it moves diagonally upward on the diagram with increase in speed. The vertical component is due to the $U^6$ rule, while the horizontal component is due to the dependence of $f$ on $U$ in the Strouhal number.

If the original spectrum is heavily weighted toward the low frequencies, more energy is added to the frequency band at the low end than is removed at the high end as the speed increases. As a result, the deduced speed dependence will be greater than $U^6$. Similarly, if the spectrum is weighted toward the higher frequencies, the speed dependence will appear to be less than $U^6$. Since dipole sources do not radiate well at low frequencies, it is likely that the dependence will be less than $U^6$, which is what has been found experimentally.
An example of this type of deviation is shown in Figures 6-4 and 6-5. The first shows a peaked spectrum with a slope of -3dB/octave on either side of the peak. At low speed, the peak is out of the measurement band and the minus slope part of the spectrum encompasses the entire frequency range. At high speed, the peak has moved into the measured frequency band, introducing lower levels at the lower frequencies. For this example, the speed dependence was calculated to be \( U^{5.4} \) as opposed to \( U^6 \).

**Key Point:** The broadest possible measurement bandwidth is the best approach, particularly for large machines.

### 6.4.2 Strouhal Number and Source Strength

Each of the source types has a characteristic Strouhal number spectrum based on the specific application. How the actual frequency spectrum changes with speed is shown below. Data suggest that the Strouhal number is a weak function of the Reynolds number, say

\[
S = S_0 \Re^{-1/n}
\]

where \( n \) might be an integer. The speed exponent of the sound power would be reduced to that shown in Table 6-4.

<table>
<thead>
<tr>
<th>Source Type</th>
<th>Monopole</th>
<th>Dipole</th>
<th>Quadrupole</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relation to Power</td>
<td>( S^2 )</td>
<td>( S^2 )</td>
<td>( S^4 )</td>
</tr>
<tr>
<td>Exponent 1/n</td>
<td>( U^{1-2/n} )</td>
<td>( U^{6-2/n} )</td>
<td>( U^{8-4/n} )</td>
</tr>
<tr>
<td>Exponent 1/5</td>
<td>( U^{5.6} )</td>
<td>( U^{5.6} )</td>
<td>( U^{7.2} )</td>
</tr>
<tr>
<td>Theory</td>
<td>( U^4 )</td>
<td>( U^6 )</td>
<td>( U^8 )</td>
</tr>
</tbody>
</table>

*Table 6-4. Modified speed dependence based on Reynolds number dependence.*
The dimensionless source strengths for each type may have a dependence on speed. Consider that the source strength has a weak negative dependence on Reynolds number, such as shown in the equations on the right. Table 6-5 shows the extreme result when both the Strouhal number and the dimensionless source strength have weak negative dependence on Reynolds number.

<table>
<thead>
<tr>
<th>Source Type</th>
<th>Monopole</th>
<th>Dipole</th>
<th>Quadrupole</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponent 1/5</td>
<td>$U^{1/2}$</td>
<td>$U^{3/2}$</td>
<td>$U^{6/3}$</td>
</tr>
</tbody>
</table>

Table 6-5. Modified speed dependence based on Strouhal and source strength dependence on Reynolds number.

### 6.4.3 Near Field

Consider a dipole source with broadband sound pressure spectrum as it changes with speed. A band that was once in the near field (kr<1) may move to a more efficient frequency based on the changing radiation impedance (Eq. 4.14). This adds more sound as the speed is increased. The mean square sound pressure is given in the equation on the right. If dynamic similarity holds, Eq 4.9 is relevant \{A.2.2.7\} so the force has the speed dependence shown on the right. As an illustration, a force spectrum (with a -3 dB/octave up to the maximum and a – 9 dB/octave slope above the maximum) was used to model this effect. The speed was doubled in three steps from 50 ft/sec to 400 ft/sec. The resultant spectrum for each speed is shown in Figure 6-6. The spectrum contour changes dramatically with speed.

![Fig. 6-6. Near field influence on spectrum.](image1)

![Fig. 6-7. Near field influence on speed dependence.](image2)

The overall sound pressure level was calculated for each speed and is compared with the theoretical dipole model of $U^6$ in Figure 6-7. The calculated speed dependence was $U^{6.2}$. This is
a case of speed dependence being higher than the theoretical model when significant low frequency sound exists.

6.4.4 Change of Source Type

A classic example of change of source is the flow from an exhaust pipe, such as the example shown in Figure 6-2. At low speeds the fluctuating mass flow rate (monopole) dominates, changing to lip sound (radial dipole, \{4.9.5\}) as the boundary layer of the flow controls the sound making, and then changing to jet sound (quadrupole, \{5.3.2\}). Figure 6-8 shows the speed dependence as might be found for such a situation. The curvature of the contour is the best clue that different source types are dominant. If such smooth data were obtainable, it would be possible to fit the theoretical speed dependences of each source type to get a best fit. Unfortunately, based on the previous sections, it is highly unlikely that the theoretical dependences would be close fits.

![Fig. 6-8. Speed dependence with change of source type.](image)
6.4.5 Frequency Weighting

All of the previous material is based on linear frequency weighting. How does other weighting methods influence speed dependence? In one study, a sound monitor was placed in a remote desert area (a national monument) to determine the impact of human activities on the sound environment. Over 1700 samples of the hourly maximum air speed and maximum A-weighted sound level were taken; the results are shown in Figure 6-9. It is clear than human activity impinged on this area, the highest levels likely due to aircraft or nearby all-terrain vehicles. The boundary on the right is of most interest and most data fall along this boundary. The limit is set by wind generated noise. The heavy dashed line represents the \( U^6 \) speed dependence for an unweighted sound spectrum. That dependence was chosen since air flow over the monitor, the wind screen, nearby branches and rock structures, is likely to be of dipole origin. The general trend of this dependence is followed, but not exactly. The wind noise spectrum is composed of several components, each of which has a different spectrum and level. Depending on the overall spectrum, application of A-weighting to it at various speeds can cause the speed dependence contour to have either positive or negative curvature as well as different overall slope. A model was created using a haystack sound spectrum whose maximum was located at very low level and frequency at low air speeds and moved upward in frequency following Strouhal scaling and upward in level following dipole scaling, similar to that in \{6.4.1\}. The spectra were then A-weighted. The thin line represents the resultant speed dependence caused by the application of weighting; it agrees with the data more closely.

**Key point:** Speed dependence must be determined with flat weighted spectra.

![Fig. 6-9. Desert monitor maximum A-weighted sound levels.](image-url)
6.5 Influence of the Human Range of Hearing

The previous chapters looked at the changes in sound level and frequency in objective terms using model sources and the similarity rules. Since primary interest is in how humans respond to such sounds, Figure 6-10 shows the Loudness contours of Figure 1-9 overlaid with representative speed dependence levels for the three types of Category II sources. For illustrative purposes, each source type has an initial level near 26 dB at 100 Hz, just barely outside the range of human hearing. Each dashed line represents the objective level and frequency increase for each of the three sources over a ten-to-one speed range. Visualize sliding these lines around the graph to show the wide variety of subjective responses that can be obtained when speed is either increased or reduced.

As an example, calculations were made for a doubling of speed starting at 60 dB for the three source types at three starting frequencies. The results are shown in Table 6-6. The objective level increase was the same for each frequency range, but the loudness increase depended strongly on it. The worst case was that for low starting frequencies where the subjective increase far outweighed the objective increase; this might be the case for many large, low frequency machines. At the mid frequencies, the contours are relatively flat so the subjective increase was similar to the objective increase. As the frequency range approaches the most sensitive range of the ear, the subjective increase was slightly greater than the objective increase. For starting frequencies above 4000 Hz, the subjective change was either about the same as the objective change or negative for the monopole source.

Fig. 6-10 Speed dependence lines superimposed on the equal loudness contours.
Paying attention to the response of listeners can prevent moving the sound of a machine into an objectionable area of listening. In a similar way, reduction of objective noise from a machine may provide more subjective benefits than expected.

<table>
<thead>
<tr>
<th>Frequency Range</th>
<th>Measure</th>
<th>Monopole</th>
<th>Dipole</th>
<th>Quadrupole</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>Linear</td>
<td>12</td>
<td>18</td>
<td>24</td>
</tr>
<tr>
<td>50 to 100</td>
<td>Phons</td>
<td>32</td>
<td>41</td>
<td>50</td>
</tr>
<tr>
<td>500 to 1000</td>
<td>Phons</td>
<td>8</td>
<td>13</td>
<td>20</td>
</tr>
<tr>
<td>1000 to 2000</td>
<td>Phons</td>
<td>15</td>
<td>22</td>
<td>29</td>
</tr>
</tbody>
</table>

Table 6-6. Increases due to a doubling of speed for pure tones from the three sources.

Consider the influence of broadband random sound on a listener. The graph of Figure 1-10 is particularly relevant to Category I sound sources. Since the spectrum may cover most of the graph, a graphical presentation is not helpful. As an example, a broadband spectrum with an approximate 4 dB/octave rolloff was used for the calculation. Each one-third octave band sound level was converted to a subjective value of Noy ($N_i$) by use of the contours in Figure 1-10. These values were summed to overall noisiness Noys using the first of Eqs. 6.2. The value of $K$ was set to 0.15 to represent one third octave bandwidths. $N_m$ is the maximum Noy value and $N_i$ is the value in each band. The overall noisiness is converted to PNdB with use of the second of Eqs. 6.2.

$$\text{Noys} = N_m (1 - K) + K \sum_{i=1}^{n} N_i$$

$$\text{PNdB} = 40 + 33 \log_{10} (\text{Noys})$$

The results are shown in Table 6-7. The Perceived Noise Levels are compared with the usual dB(A) evaluation for three speeds. It is clear that Perceived Noise ratings result in a higher subjective noise rating than A-Weighted levels.

<table>
<thead>
<tr>
<th></th>
<th>U</th>
<th>2U</th>
<th>4U</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Monopole</strong></td>
<td>dB(A)</td>
<td>46.9</td>
<td>62.7</td>
</tr>
<tr>
<td></td>
<td>PNdB</td>
<td>48.9</td>
<td>74.5</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td>2.0</td>
<td>11.0</td>
</tr>
<tr>
<td><strong>Dipole</strong></td>
<td>dB(A)</td>
<td>46.9</td>
<td>68.7</td>
</tr>
<tr>
<td></td>
<td>PNdB</td>
<td>46.9</td>
<td>80.7</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td>2.0</td>
<td>12.0</td>
</tr>
</tbody>
</table>

Table 6-7. Comparison of PNdB and dB(A) ratings for a typical noise spectrum over a 4 to 1 speed range.

**Key Points:** Both the spectrum contour and the frequency range of a broadband sound spectrum are important in evaluating the impact of the sound on listeners. Speed increases for spectra that
have a maximum at low frequencies have noisiness increases that exceed the increase in A-weighted level. Speed decreases for spectra that have a maximum at mid-frequencies have noisiness decreases that exceed the decrease in A-weighted level.
Appendix A
General Information

A.1 Useful Data

A.1.1 Conversion Factors

Acoustics uses the metric system exclusively, but unfortunately American engineering still uses the English system (which the English have abandoned—except vehicle speeds and distances in the UK). As a result unit conversions are always necessary.

\[
1 \text{ lb}_f = 4.448 \text{ newtons} = 444,800 \text{ dynes}
\]

\[
1 \text{ lb}_m = 0.4536 \text{ kilograms} = 453.6 \text{ grams} = 0.0311 \frac{\text{lb}_f - \text{sec}^2}{\text{ft}}
\]

\[
1 \text{ ft} = 0.3048 \text{ meters} = 30.48 \text{ centimeters}
\]

\[
\frac{1 \text{ ft} \cdot \text{lb}_f}{\text{sec}^2} = 1.356 \text{ Watts} = 1.356 \text{ newton-meter/sec}
\]

\[
1 \text{ lb}_f = 32.2 \frac{\text{lb}_m - \text{ft}}{\text{sec}^2}
\]

\[
1 \text{ mph} = 1.467 \text{ ft/sec} = 0.447 \text{ m/sec} = 0.869 \text{ knots}
\]

\[
1 \text{ ft}^3 = 28.33 \text{ liters} = 1728 \text{ in}^3 = 0.02832 \text{ m}^3
\]

A.1.2 Reference Quantities

A.1.2.1 Speed of Sound in Air

For Celsius the speed is in meters/second, while for Fahrenheit it is in feet/second.

\[
c_0 = 20.05 \sqrt{(273.2 + °C)}
\]

\[
c_0 = 49.03 \sqrt{(459.7 + °F)}
\]

\[
°F = \frac{9°C}{5} - 32
\]

\[
°R = 459.67 + °F
\]

\[
°K = 273.15 + °C
\]

For a standard temperature of 72°F the speed is 1130.5 ft/sec, or 344.4 m/sec.

A.1.2.2 Specific Acoustic Impedance of Air

This quantity represents the characteristics of the source surroundings and so is implicit in all acoustic equations. It is the ratio of the force at a point in a free progressive wave to the response (velocity) at the same point. By its nature it is resistive.
\[ Z_0 = \frac{p}{u} = \rho_0 c_0 \quad \text{N-sec/m}^3 \text{(mks Rayls)} \]

For 72°F and normal atmospheric conditions, \( Z_0 = 406 \text{ Rayls} \). Since English units are used here, it is necessary to convert this value.

\[ Z_0 = \rho_0 c_0 = 406 \frac{N-\text{sec}}{m^3} \times \frac{1 \text{ lb}_{f}}{4.448 \text{N}} \times \left( \frac{1 \text{m}}{3.281 \text{ft}} \right)^3 = 2.584 \frac{\text{lb}_{f} - \text{sec}}{\text{ft}^3} = 83.2 \frac{\text{lb}_{m}}{\text{ft}^2 - \text{sec}} \]

The latter value is useful for known mass flow rates.

A.1.2.3 Density of Air at 70° F

\[ \rho_0 = 0.0750 \frac{\text{lb}_{m}}{\text{ft}^3} = 0.00233 \frac{\text{lb}_{f} - \text{sec}^2}{\text{ft}^4} \]

A.1.2.4 Viscosity of Air at 70° F

Dynamic \[ \mu = 1.22 \times 10^{-5} \frac{\text{lb}_{f}}{\text{ft} - \text{sec}} = 3.79 \times 10^{-7} \frac{\text{lb}_{f} - \text{sec}}{\text{ft}^2} \]

Kinematic \[ \nu = \frac{\mu}{\rho} = 1.63 \times 10^{-4} \frac{\text{ft}^2}{\text{sec}} \]

A.1.2.5 Gas Constant of Air

\[ \gamma = 1.4, \quad R = 53.3 \frac{\text{lb}_{f} - \text{ft}}{\text{lb}_{m} - \text{R}} \]

A.1.2.6 Two Important Constants

\[ \pi = 3.1415926535 \ldots \]
\[ e = 2.7182818284 \ldots \]

A.1.3 Level References

Sound Power \[ W_R = 10^{-12} \quad \text{Watts} \]

Sound Pressure \[ P_R = 2 \times 10^{-5} \quad \text{N/m}^2 \quad P_R = 2.90 \times 10^{-9} = \text{lb} / \text{in}^2 \]

Intensity \[ I_R = 10^{-12} \quad \text{Watts/m}^2 \]

A.1.4 Levels

Sound Power Level \[ L_W = 10 \log_{10} \left( \frac{W}{W_R} \right) \]

Sound Pressure Level \[ L_p = 10 \log_{10} \left( \frac{p^2}{P_R^2} \right) = 20 \log_{10} \left( \frac{p_{\text{rms}}}{P_R} \right) \]
A.2 Units and Dimensions

A.2.1 Dimensional Variables

Much of the analysis of sound problems depends on the correct determination of the dimensions of variables. Engineering units are susceptible to large errors, particularly conversion of mass to force. It is always necessary to make sure the dimensions of each term of an equation are the same. An example of some equations that are used elsewhere in this monograph is shown in Eqs. A.1. The exponent in the expression of velocity potential must be dimensionless, so each of the component terms must also be dimensionless. The pressure and velocity are related to the velocity potential \{A.3.1\}; the dimensions are indicated to the right of the expression. Solving for A yields differing results, until the conversion from mass units to force units is done. Checking dimensional consistency is the best way to find missing values. The number 32.2 used to convert mass to force is large in the engineering system, which is one reason why the metric system has been introduced. Unfortunately, acoustics, based on the much more logical metric system, is embedded in a sea of engineering units in this country.

Since dimensional consistency is important, the dimensions of the variables used in this monograph are given below. These are useful to check the dimensions of any equation.

- Time \( T \)
- Length \( L \)
- Force \( F \)
- Mass \( M \)
- Frequency \( 1/T \)
- Wave number \( L^{-1} \)
- Speed \( L/T \)
- Velocity Potential \( L^2/T \)
- Pressure \( F/L^2 \)
- Dynamic Viscosity \( FT/L^2 \)
- Kinematic Viscosity \( L^2/T \)
- Impedance \( FT/L^3 \)
- Density \( FT^2/L^4=M/L^3 \)
- Volumetric Flow Rate \( L^3/T \)
- Force Moment \( FL \)
- Intensity \( F/LT \)
- Power \( FL/T=ML^2/T^3 \)
A.2.2 Dimensionless Variables

Nature knows nothing of the dimension system we use; it is concerned only with the ratios of things. Dimensional similarity refers to the ratio of dimensions preserved when an object’s size is changed. Dynamic similarity refers to the ratio of forces preserved when the forces are changed. Dimensionless variables are an extremely useful way to take changes into account. If the dimensionless variables do not change when the dimensional variables change, similarity is achieved and tests in one situation can be extrapolated to another. This is quite important in sound generation. The major dimensionless variables associated with fluid mechanics are derived in \{D.2\}. They imply a choice of two characteristic dimensional variables, a speed \(U\) and a length \(L\) that are used to create the dimensionless variables. They relate to both the forces and geometry involved. The art is in choosing these two important variables. All the examples in this monograph use this art to show how sound problems can be approached in new situations.

A.2.2.1 Strouhal number

\[ St = \frac{fL}{U} \]

The ratio of unsteady inertial forces to steady inertial forces. The number is named in honor of Vincenz Strouhal (1850-1902) who first deduced the relationship between the vortex shedding frequency around a cylinder, the cylinder diameter \(L\), and the speed of the flow over it \(U\). The number was found to be virtually constant from a Reynolds number of 500 to 200000. This number permits relationships to be developed between different sizes, speeds, and time variables rates. This number is developed from the mass continuity equation \{D.2\}. It can be considered to be a hydrodynamic Strouhal number since the characteristic speed is that of the fluid. The numerical value of the Strouhal number would better match the other variables if the frequency were expressed in radians/second \{A.2.2\}.

A.2.2.2 Helmholtz number

\[ He = kL = \frac{\omega L}{c_0} = \frac{2\pi fL}{c_0} = \frac{2\pi L}{\lambda} = 2\pi St_a \]

The characteristic length \(L\) is expressed in sound wavelengths. The characteristic speed is that of sound. The variable \(k\) is called the wave number despite the fact that its dimensions are \(L^{-1}\). The wave number is integral to the mathematical development of sound sources. The attribution to Hermann Helmholtz (1821-1894) is that the characteristic length \(L\) was the radius of a tube and the specific value of the number referred to specific frequencies of a tube resonance. Note the similarity of the Strouhal number to this number; the difference being the characteristic speed. Since the Strouhal number is commonly used, the relationship above is also expressed as an acoustical Strouhal number where the characteristic speed is that of sound. When used in this form, the subscript \(a\) will be used.

A.2.2.3 Mach number

\[ Ma = \frac{U}{c_0} \]

The ratio of the steady speed to the speed of sound. The number is named in honor of Ernst Mach (1838-1916) who first studied (among other things) supersonic motion and the shock waves generated. He developed a method to see the otherwise invisible shock structure. This number permits identification of the separation point between incompressible and compressible flow. This variable is developed from the momentum equation.
A.2.2.4 Reynolds number

\[ \text{Re} = \frac{UL}{v} \]

The ratio of the steady inertial forces to the steady viscous forces. The number is named in honor of Osborne Reynolds (1842-1912), an engineer who did pioneering studies on the transition of laminar to turbulent flow in pipes. This number permits relationships to be developed between different sizes, speeds and fluids. This variable is developed from the momentum equation.

A.2.2.5 Rossby number

\[ \text{Ro} = \frac{U}{f_0L} \]

The ratio of linear velocity to tangential velocity for swirl flows. The frequency is characteristic of the rotation rate of the flow. The number is named in honor of Carl-Gustav Rossby (1898-1957), a meteorologist who first described the large scale motions of the atmosphere in terms of fluid mechanics. He described the jet stream, and his number was first used to describe the motion associated with the coriolis force in the atmosphere. This variable is developed elsewhere from the equations of motion in curvilinear coordinates.

A.2.2.6 Frequency

\[ \omega t = 2\pi ft = \frac{2\pi c_0t}{\lambda} = k c_0 t \]

It can be considered a dimensionless frequency or the ratio of the distance sound travels in time \( t \) to the sound wavelength.

A.2.2.7 Force

\[ \hat{F} = \frac{F}{\rho_0 U L^2} \]

The ratio of the actual dynamic force to the steady momentum.

A.2.2.8 Force Moment

\[ \hat{\tau} = \frac{\tau}{\rho_0 U^2 L^3} \]

The ratio of the dynamic stress moments to the steady stress moments.

A.2.2.9 Volumetric Flow Rate

\[ \hat{Q} = \frac{Q}{UL^2} \]

The ratio of the dynamic volumetric flow rate to the steady volumetric flow rate.

A.2.2.10 Power

\[ \hat{W} = \frac{W}{\rho_0 U^3 L^2} \]

The ratio of the dynamic (sound) power to the steady power.
A.3 Physical Variables Derived from Velocity Potential

A.3.1 In The Time Domain

The velocity potential in spherical coordinates is \( \phi(r, \theta, \psi, t) \). The following physical acoustical variables can be derived shown in Eqs. A.2.. The dimensions of each variable are also shown.

\[
\begin{align*}
\rho_0 \frac{\partial \phi}{\partial t} &= \frac{F}{L} \\
p(r, \theta, \psi, t) &= \rho_0 \frac{\partial \phi}{\partial t} - \frac{F}{L} \\
u_r(r, \theta, \psi, t) &= -\frac{\partial \phi}{\partial r} \frac{L}{T} \\
u_\theta(r, \theta, \psi, t) &= \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \theta} \frac{L}{T} \\
u_\psi(r, \theta, \psi, t) &= -\frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \psi} \frac{L}{T} \\
s(r, \theta, \psi, t) &= \frac{1}{c_0^2} \frac{\partial \phi}{\partial t} = \frac{\rho - \rho_0}{\rho_0} \\
T(r, \theta, \psi, t) &= \frac{1}{R} \left( \frac{\gamma - 1}{\gamma} \right) \frac{\partial \phi}{\partial t} \\
I_r(r, \theta, \psi) &= \rho \cdot u_r^* = -\rho_0 \frac{\partial \phi}{\partial t} \frac{\partial \phi^*}{\partial r} \\
W &= \int_0^{2\pi} \int_0^\infty I_r(r, \theta, \psi) \rho_0^2 \sin \theta d\theta d\psi
\end{align*}
\]

The next to last term is the density fluctuation and is called condensation, note that it is dimensionless. It is expressed in lower case to distinguish it from the Strouhal number. Note that pressure, density and temperature are all in phase as might be expected with fluid compression.

A.3.2 In the Frequency Domain

The velocity potential in spherical coordinates is \( \Phi(r, \theta, \psi, \omega) \). It is related to the time domain through the relationship

\[
\mathcal{F}[\varphi] = \Phi(r, \theta, \psi, \omega) = \int_{-\infty}^{\infty} \varphi(r, \theta, \psi, \tau) e^{-i\omega \tau} d\tau
\]

In a similar manner, the following physical variables can be derived from it.
A.4 Fourier Series

A.4.1 Real Fourier Series

Simple harmonic motion is restricted to one frequency. Complex harmonic (periodic) motion contains several frequencies. Examples are: square and triangular waves. To handle them, one must expand the time history by using the Fourier series expansion of that function. The basic equations are:

\[ p(r, \theta, \psi, \omega) = \rho_0 \mathfrak{A} \left[ \frac{\partial \varphi}{\partial t} \right] = i \omega \rho_0 \Phi \]

\[ u_r (r, \theta, \psi, \omega) = -\frac{\partial \Phi}{\partial r} \]

\[ u_\theta (r, \theta, \psi, \omega) = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \]

\[ u_\psi (r, \theta, \psi, \omega) = -\frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \psi} \]

\[ s(r, \theta, \psi, \omega) = \frac{ik \Phi}{c_0} \]

\[ T(r, \theta, \psi, \omega) = \frac{i \omega}{R} \left( \frac{\gamma - 1}{\gamma} \right) \Phi \]

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n \pi x}{T} + b_n \sin \frac{n \pi x}{T} \right) \]

\[ a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos \left( \frac{n \pi x}{T} \right) dx \]

\[ b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin \left( \frac{n \pi x}{T} \right) dx \]

\( a_0 \) is the steady value. It is generally set to zero for acoustical applications. \( T \) is the half period.
Example

A square wave of half-amplitude \( A \) has a period \( 2T \) of two around a value of 0. Thus the variable \( T \) is 1, and \( a_n \) is zero. \( f(x) = -A \) from -1 to 0 and \( f(x) = A \) from 0 to 1. The integrals are shown in the next set of equations.

\[
\begin{align*}
a_n &= -\int_{-1}^{0} A \cos(n\pi x)\,dx + \int_{0}^{1} A \cos(n\pi x)\,dx = 0 \\
b_n &= -\int_{-1}^{0} A \sin(n\pi x)\,dx + \int_{0}^{1} A \sin(n\pi x)\,dx = \frac{A}{n\pi} \left[ 1 - \cos(n\pi) \right] \\
b_n^* &= \frac{4A}{n\pi} \\
b_n &= 0
\end{align*}
\]

Odd values of \( n \) have finite values.

\[ f(x) = 4A \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi} \]

A square wave was constructed of a sinusoid of period 2; the summation of its odd harmonics are shown in the figures below. There is always an overshoot at the transition with a finite number of terms. Albert Michelson (1852-1931) noticed this but attributed it to error. J. Willard Gibbs showed that it was an artifact of the Fourier series. It is based on approximating a discontinuous function with a finite series of continuous functions. It is now call the Gibbs phenomenon. Figure A-1 shows a 9 term fit to a square wave. Figure A-2 shows the Gibbs phenomenon more clearly with 105 terms.

![Synthesized Square Wave of Period 2](image1)

*Fig. A-1. A 9 term fit of Fourier series to a square wave.*

![Figure A-2. A 105 term fit of Fourier series to a square wave.](image2)
A.4.2 Fourier Integrals

For functions that are more complex, such as random sound or transient motions, it is necessary to work in the frequency domain. That is the purpose of the Fourier Integral. There are two equations that act as transforms from one form to another, one in the time domain and the other in the frequency domain. They are called Fourier Transform pairs. In complex form they are:

\[ F(\omega) = \mathcal{F} \left[ f(t) \right] = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega \tau} d\tau \]

\[ f(t) = \mathcal{F} \left[ F(\omega) \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \]

The second form of the equations is shorthand to describe that a Fourier transform has occurred without writing out the equation itself. The value of this transform can be seen if the simple mass, spring, oscillator equation is used; it is

\[ M\ddot{x}(t) + Kx(t) = f(t) \]

Each dot represents a differentiation of the displacement \( x \) and \( f \) represents an arbitrary force. If the force is considered harmonic, the solution is straightforward, but since it is arbitrary, it is not. Using the transform, the equation form reduces to one in which frequency spectrum replaces the time.

\[ M\mathcal{F}\ddot{x}(t) + K\mathcal{F}x(t) = \mathcal{F}f(t) \]

We are now dealing with the frequency spectrum of the force. If the derivatives of the displacement can be determined, the problem can be solved. See below

The real form of the integral is

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ A(\omega) \cos \omega t + B(\omega) \sin \omega t \right] d\omega \]

\[ A(\omega) = \int_{-\infty}^{\infty} f(\tau) \cos \omega \tau d\tau \]

\[ B(\omega) = \int_{-\infty}^{\infty} f(\tau) \sin \omega \tau d\tau \]

A.5 Complex Fourier Transforms

A.5.1 Stationary

\[ f(t) = \frac{1}{\pi} \int_{0}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{\pi} \int_{0}^{\infty} F(\omega) e^{-i\omega t} d\omega \]

\[ \mathcal{F} \left[ f(t) \right] = F(\omega) = \left| F(\omega) \right| e^{i\theta} = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega \tau} d\tau \]
A.5.2 Transient

\[ f(t) = 2 \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \]

\[ F(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega \tau} d\tau \]

A.6 Cosine Fourier Transforms

A.6.1 Stationary

\[ f(t) = \frac{1}{\pi} \int_{0}^{\infty} F_{c}(\omega) \cos \omega t d\omega \]

\[ F_{c}(\omega) = \left| F_{c}(\omega) \right| \cos \theta = \int_{-\infty}^{\infty} f(\tau) \cos \omega \tau d\tau \]

A.6.2 Transient

\[ f(t) = 2 \int_{0}^{\infty} F_{c}(\omega) \cos \omega t d\omega \]

\[ F_{c}(\omega) = \int_{0}^{\infty} f(\tau) \cos \omega \tau d\tau \]

A.7 Sine Fourier Transforms

A.7.1 Stationary

\[ f(t) = \frac{1}{\pi} \int_{0}^{\infty} F_{s}(\omega) \sin \omega t d\omega \]

\[ F_{s}(\omega) = \left| F_{s}(\omega) \right| \sin \theta = \int_{-\infty}^{\infty} f(\tau) \sin \omega \tau d\tau \]

A.7.2 Transient

\[ f(t) = 2 \int_{0}^{\infty} F_{s}(\omega) \sin \omega t d\omega \]

\[ F_{s}(\omega) = \int_{0}^{\infty} f(\tau) \sin \omega \tau d\tau \]
A.8 Transform Relationships

\[ F(\omega) = F_\text{s}(\omega) - iF_\text{c}(\omega) \]

\[ |F(\omega)|^2 = F(\omega)F^*(\omega)^* = F_\text{s}(\omega)^2 + F_\text{c}(\omega)^2 \]

\[ \tan\theta = \frac{F_\text{s}(\omega)}{F_\text{c}(\omega)} \]

\[ \mathcal{Z}\left[ \frac{df(t)}{dt} \right] = -f(\beta) + i\omega F(\omega) \]

\[ \mathcal{Z}\left[ \frac{d^2 f(t)}{dt^2} \right] = -\frac{df(\beta)}{dt} - i\omega f(\beta) - \omega^2 F(\omega) \]

\[ f(\beta) = 0, \frac{df(\beta)}{dt} = 0, \text{stationary} \]

\[ f(\beta) = f(0), \frac{df(\beta)}{dt} = \frac{df(0)}{dt}, \text{transient} \]

\[ \mathcal{Z}\left[ f(t + \tau) \right] = e^{-i\omega \tau} F(\omega) \]

The equations above are limited to positive frequencies, since it is not possible to distinguish positive from negative. The factor of 2 appears in the cosine and sine integrands to make them even functions so that the time function is real. For the transient cases, the integrals have been restricted to positive times resulting in another factor of 2.

A.8.1. Square Pulse

A pulse of amplitude \( H \) occurs for a time \( T \).

\[ F(\omega) = H \int_0^T e^{-i\omega \tau} d\tau = \frac{iH}{\omega} \left[ e^{-i\omega T} - 1 \right] \]

\[ F(\omega) \cdot F^*(\omega) = \frac{2H^2}{\omega^2} \left[ 1 - \cos \omega T \right] = \frac{2H^2}{\omega_\text{c}^2} \left[ 1 - \cos \pi \alpha \right] \]

The equations below define a ratio of frequencies to the fundamental periodicity of the pulse.

\[ \omega_\text{c} T = \pi \]

\[ \alpha = \frac{\omega}{\omega_\text{c}} \]

Figure A-3 show the relative levels as a function of frequency in terms of the ratio \( \alpha \). Note that minima occur at even multiples of the base period.
Fig. A-3. The frequency spectrum of a square pulse.
Appendix B

Geometry and Complex Operations

B.1 Spherical Geometry and the Wave Equation

The equations of fluid motion are embodied in the equations of mass, momentum and thermodynamics. One result of linearization is the wave equation shown below in spherical coordinates. This equation suggests that wave motion must be embedded in the more complex motions of fluids and care must be taken to define the limitations of the equation and the approximations needed for its validity. This is done in Appendix D. The equation describes only wave motion and has no indication of the source of the motion. Thus it must be applied away from any source regions. Some texts include a source term, but it will be shown that much can be learned without it. All fluids have viscosity, but the wave equation excludes it.

The wave equation in spherical coordinates is

\[ \Box^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \psi^2} - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \]  

(B.1)

where \( r \) is the radial coordinate, \( \theta \) and \( \psi \) are polar coordinates. The relation to Cartesian coordinates is shown in the equations below. The element of area on the surrounding spherical surface is shown in Figure B-1. To integrate over the sphere, \( \theta \) must vary from 0 to \( \pi \), and \( \psi \) must vary from 0 to \( 2\pi \).

Eq. B.1 is best expressed in terms of a velocity potential \( \alpha.6.1 \) from which the physical variables can be readily derived. These relationships are given below.

\[
\begin{align*}
p &= \rho_o \frac{\partial \phi}{\partial t} \\
u_r &= -\frac{\partial \phi}{\partial r}, \quad u_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad u_\psi = -\frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \psi} \\
s &= \frac{1}{c_0^2} \frac{\partial \phi}{\partial t} \\
x &= r \sin \theta \cos \psi \\
y &= r \sin \theta \sin \psi \\
z &= r \cos \theta \\
dA &= r^2 \sin \theta \, d\theta \, d\psi
\end{align*}
\]

(B.2)

\[ \text{Fig. B-1. Spherical coordinates.} \]
B.2 Complex Notation

B.2.1 The Constant e

The constant $e$ is used in all areas of mathematics and is very useful in the complex notation used for sound generation. An early form is given below.

$$e^{i\pi} + 1 = 0$$

_Euler’s (Napier’s) Identity: “The most beautiful theorem in math”_  
It ties the two most used irrational numbers to the imaginary operator and the integers all in one relationship.

But what is the meaning of $e$? There are as many ways of defining it as there are ways to use it. One is to define it as the area under the hyperbolic curve shown in Figure B.2.

![Fig. B-2 The value of $e$ as an integral limit.](image1)

The integral for it is given in the first equation to the right. It is the value of the upper limit that makes the area under the curve equal to one. It is the base of the natural (Naperian) logarithms. It can be defined as an infinite series as shown in the second equation. The third expression is a generalization of the identity shown at the section beginning that covers all the zeroes. The number ($e=2.7182818284…$) is referred to as Euler’s number since it was the Swiss mathematician Leonard Euler (1707-1783) that made use of it. It is also referred to as Napier’s number since John Napier used it as the base of logarithms. Complex notation makes extensive use of this number.

![Fig. B-3 Complex coordinates.](image2)
B.2.2 Simple Harmonic Motion

Simple harmonic motion is best described using complex notation. Figure B-3 shows what is called the real and imaginary axes. Any point on the plane must be described by two numbers. The complex number, 3+i2, is the end of the arrow in the figure. The letter i (j is used in some engineering documents) is added to indicate the imaginary axis.

It is important to note that \( i = \sqrt{-1} \) has two roots, and it was first used by Gerolomo Cardano (1501-1575), who first described typhoid fever. One must stand in awe at the cleverness of the first person to express complex geometry in terms of an irrational number raised to the power of an imaginary number, e.g., \( e^{i\pi} \).

Typical descriptors in polar coordinates are shown in Figure B-4. The arrow can be described by the linear dimensions or the radius and angle. Putting these two figures together we get the following relationships.

\[
R = a + ib = R (\cos \theta + i \sin \theta) = \sqrt{a^2 + b^2} e^{i \tan^{-1}(\frac{b}{a})} = R e^{i\theta}
\]

\[
(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta = e^{i n\theta}
\]  \hfill (B.3)

The latter formula comes from a theorem by Abraham de Moivre (1667-1754) through use of infinite series expansions.

The main advantage of using the exponential form is ease of manipulation. Consider the addition of two complex numbers, \( x=a+ib \) and \( y=c+id \). The rules of algebra, give the simple result \( x+y=(a+c)+i(b+d) \).

Consider multiplication of two complex numbers. When doing so, \( i^2 \) is encountered. Other manipulations of the imaginary number are often needed. Table B-1 below shows a table of results for common operations with that number. The equation for one calculation is shown on the right of the table.

\[
i = e^{\frac{i\pi}{2}}
\]

\[
i^2 = e^{i\pi} = -1
\]

Table B-1. Manipulating the square root of minus one.

<table>
<thead>
<tr>
<th>( i^0 )</th>
<th>( i^1 )</th>
<th>( i^2 )</th>
<th>( i^3 )</th>
<th>( i^4 )</th>
<th>( i^{-1} )</th>
<th>( i^{-2} )</th>
<th>( i^{-3} )</th>
<th>( i^{-4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>i</td>
<td>-1</td>
<td>-i</td>
<td>1</td>
<td>-i</td>
<td>-1</td>
<td>i</td>
<td>1</td>
</tr>
</tbody>
</table>

Eqs B.3 can be used to convert each variable to polar form as shown on the right. Carrying out the multiplication in polar form gives a simpler result. The arithmetic approach yields \( xy=(ac-bd)+i(bc+ad) \). This form is difficult to interpret when dealing with frequencies and phases, so polar notation is used throughout this monograph.

There is one other complex number operation that is useful in sound analyses: the complex conjugate and is often denoted by an exponent *; the sign of the imaginary operator is reversed. The sound pressure can be described in complex notation, but to calculate mean square pressure, the equation must be reduced to its real component, which is easy with complex notation as shown on the right.

\[
x = a + ib = Ae^{i\theta}
\]

\[
y = c + id = Be^{i\phi}
\]

\[
x + y = Ae^{i\theta} + Be^{i\phi}
\]

\[
xy = AB e^{i(\theta+\phi)}
\]

\[
p = a + ib = P e^{i\theta}
\]

\[
p^* = a - ib = P e^{-i\theta}
\]

\[
p \cdot p^* = P^2 = a^2 + b^2
\]
B.3 The Power of Complex Notation

To develop solutions for dynamics problems it is often necessary to start from first principles. Unfortunately, there is very little guidance on how this is done. In this section, the classic vibrating mass-spring problem is solved using first principles (almost), and then more modern methods.

B.3.1 The Original Method

There is a mass suspended from a spring and damper (Figure B-5) and the question is: What happens when the mass is disturbed? The addition of a damping mechanism is not needed for this example. The answer to this question can be found in textbooks [6].

The static equation and the equation of motion using Newton’s law are given below. The subscript zero refers to the static position when no spring force is exerted ($x=x_0$). The subscript one refers to the static position with the weight attached.

\[ F = Mg - K(x + x_1 - x_0) \quad (B.4) \]

\[ F = M \ddot{\frac{\partial^2 x}{\partial t^2}} \]

The solution for displacement will be developed by use of power series expansions, just as did Friedrich Bessel (1784-1846) to laboriously calculate by hand each of the terms of the cylindrical functions originated by Daniel Bernoulli (1700-1782). These functions are now known as Bessel functions. The equations are

\[ x = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \ldots \]

\[ M \ddot{x} + Kx = Mg - K(x_1 - x_0) \]

\[ Mg - K(x_1 - x_0) = 0 \]

\[ \ddot{x} + \frac{K}{M} x = 0 \quad (B.5) \]

\[ a_2 = -\frac{K a_0}{2M}, a_3 = -\frac{K a_1}{6M}, a_4 = -\frac{K a_2}{12M} = \left( \frac{K}{M} \right)^2 \frac{a_0}{24} \]

\[ x = a_0 \left[ 1 - \left( \frac{K}{M} \right)^2 \frac{t^2}{2!} + \left( \frac{K}{M} \right)^2 \frac{t^4}{4!} - \ldots \right] + a_1 \left[ t - \left( \frac{K}{M} \right)^2 \frac{t^3}{3!} + \ldots \right] \]

The first equation is the series expansion of displacement $x$ at time $t$, the second is the equation of motion where the double dot represents the second time derivative. Since the left side of the equation depends on time and the right does not, they must both be independent and equal to a constant which can be set to zero (free vibration), yielding the third and fourth equations. When the displacement $x$ is differentiated twice and substituted, and the various time
terms are equated to each other, we get relationships between the various constants as shown in
the fifth equations. Note that each of the higher constants can be related to the first two
constants. As a result, the last equation can be written as a series expansion of just two terms.
We often forget, and take for granted, the huge base of knowledge created by these early
investigators. They examined the two expansions and found that they relate to triangles, namely
the sine and cosine functions (which themselves are infinite series expansions), so the form can
be expressed as shown on the right.

Since the problem is dynamic, the square root
function must be interpreted as a frequency (dimension t\(^{-1}\)). This frequency is called the natural frequency,
which is most unnatural. Since all real systems have
damping it requires that the actual frequency always be
less than the natural frequency.

### B.3.2 The Complex Notation Method

The beauty of using complex notation, as is done throughout this monograph, is shown in
Eqs. B.6. The displacement \( x \) is expressed in complex notation.

\[
x = a_0 \cos \left( \sqrt{\frac{K}{M}} t \right) + a_1 \sqrt{\frac{M}{K}} \sin \left( \sqrt{\frac{K}{M}} t \right)
\]

\[
\omega = \sqrt{\frac{K}{M}}
\]

Since both the left and right sides of the second equation are independent, we get the
third equation directly; much easier.

### B.4 Simple Harmonic Motion

We can use complex notation to represent
simple harmonic (single frequency) motion. If we
think of the vector \( \mathbf{R} \) as rotating in a circle in the
complex plane, we can write the equations on the right
where \( \omega \) is the radian frequency, \( 2\pi f \). If the frequency is one cycle per second, the \( \mathbf{R} \) vector
rotates a complete circle once a second. This notation is very convenient for manipulating
frequency information. The \( \theta \) symbol is used to denote a phase angle which simply changes the
time of axis crossing. The use of the imaginary axis in harmonic motion is simply a clean
method of accounting for the instantaneous phase of the motion.

There are two methods of determining the “real” part of the motion (e.g., the mean square sound pressure) and they give different results. One method, shown in the first of Eqs. B.7 below, is to square the real part of the variable (it eliminates negative values) and then integrate over a period to get the desired result. Taking the square root yields root-mean-square (r.m.s.) values. The preferred method is to use complex notation as shown in the second of Eqs. B.7. The mathematical manipulation is simple, but the results are not the same as with the first method. The value of \( p_2 \) must be interpreted as the maximum amplitude of the motion while \( p_1 \)
must be interpreted as the *r.m.s. amplitude* (i.e., \( p_1 = 0.707 \times p_2 \) for simple harmonic motion). When dealing with random motion, it is not possible to define a peak amplitude at any particular frequency, but it is possible to define r.m.s. amplitudes, so the exponential form is preferred to the trigonometric form.

\[
\begin{align*}
P_2 &= p_2 \cos \omega t, P_2^2 &= p_2^2 \int_0^{\frac{2\pi}{\omega}} \cos^2 \omega t dt = \frac{p_2^2}{2} \\
P_1 &= p_1 e^{i\omega t}, P_1^2 &= P_1 \ast P_1^* = p_1 e^{i\omega t} \ast p_1 e^{-i\omega t} = p_1^2
\end{align*}
\]

(B.7)

The constants in all equations for sound intensity and sound power in this monograph will be a factor of two larger than those shown in most textbooks that use trigonometric functions to derive the equations. Typical values will be 1/4 as opposed to 1/8.
Appendix C

Random Processes

C.1 Introduction

The specification and description of random processes is a broad field of science that overlaps many engineering disciplines. The basic concepts are described here, but it is not a detailed treatment.

A random process is one in which a precise prediction of an occurrence cannot be made, only the likelihood, or probability, of its occurrence can be described. Since all that is generally available to an engineer is a time history of a process, it is necessary to develop methods with which this history can be described usefully. All theory of random processes is concerned with ensembles and for our purposes they may be considered as an infinite array of time histories (requiring an infinite number of experiments). One description of a random process is given by expectation values such as: \( E\{x(t)\} \), \( E\{x^2(t)\} \), \( E\{x(t), x(t+\tau)\} \). These might be averages of the infinity of time histories recorded by some type of meter. If the first expected value noted above is zero, the process has a zero mean (typical of sound). The second term is the expected value of the next higher moment. When all the higher moments are known the process is completely described. The third term is the expected value of a process at a later time \( \tau \). In most engineering applications, only the first two moments are generally known. For sound sources of interest here, our concern is mostly with the second moment, typically mean square sound pressure.

Since most real world situations provide only one time history, the Ergodic Hypothesis must be invoked. For practical applications, the hypothesis requires that the process be stationary (the average of a time history over one time period is the same as the average over a different time period). This should apply to all moments of the sample, but the first and second moments are only of concern, so it applies only partially. As a result, in the description of a random process we are forced to assume that any time history is representative (typical) of any others that might be taken. This process runs into trouble with transients, particularly those which occur only once. At this point the theory diverges from practice, but that does not imply that no useful information can be gained.

C.2 Correlation or Phase

Phase relationships and correlations between sources are important. The technique of “wave cancellation” is well known: set two single frequency signals out of phase and they cancel – no sound (wherever that can be accomplished spatially). Consider that many sounds have a broad-band spectrum, typically random, for which it is not possible to define a phase. For that case, the words correlation or covariance are used. To demonstrate these concepts, consider a simple set of circumstances. There are two sound pressures from separate sources, \( p_1 \) and \( p_2 \), coming together at a particular
location. What is the consequence of their interaction? The relevant relationships are shown in Eqs. C.1.

The symbol \( \Delta \) is the ratio of the sum \( p \) to \( p_1 \). The bar represents the time average and \( p_R \) is the reference pressure for sound level. If \( p_2 \) is exactly equal to \( p_1 \) (fully correlated) then \( \Delta = 6 \text{ dB} \). If \( p_2 \) has the same magnitude as \( p_1 \) but the time average of their product is zero (uncorrelated) then \( \Delta = 3 \text{ dB} \). If \( p_2 \) is exactly equal to \( -p_1 \) (negatively correlated) then \( \Delta = -\infty \text{ dB} \). Figure C-1 shows the complete range of correlation coefficients from full positive (1) to full negative (-1) for several relative magnitudes. The obvious observation is that the addition of fully correlated sources varies from 0 to 6 dB, depending on relative magnitudes, while for uncorrelated sources it varies from 0 to 3 dB. Less obvious is the fact that the addition of slightly negatively correlated source results in no increase in level, despite the doubling of the sound source; the required correlation is the relative magnitude divided by two.

If we are concerned with single frequency sources then the correlation coefficients can be replaced by phase angles between the sources. In-phase (0°) relates to a coefficient of 1, quadrature (90°) relates to a coefficient of 0 and out-of-phase (180°) relates to a coefficient of -1. The phase angle equation is \( \cos \theta = \text{relative magnitude divided by two} \). Representative phase angles to a whole degree are given in Table C-1 below.

![Addition of Partially Correlated Sources](image)

*Fig. C-1. The addition of two partially correlated sources.*

<table>
<thead>
<tr>
<th>Relative Magnitude</th>
<th>1</th>
<th>0.75</th>
<th>0.5</th>
<th>0.25</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle</td>
<td>120</td>
<td>112</td>
<td>105</td>
<td>97</td>
<td>90</td>
</tr>
</tbody>
</table>

*Table C-1. The phase angle between two sources for no level increase.*

**Key Point:** Phase relationships (or correlations) are important between sources. Since a sound field is spatial, the relationship is often purely local and not generally applicable.
C.3 Two Correlation Questions

With the full theory of sound generation [12], an integral over a distribution of sources is performed. It completely accounts for the phase relationships, or correlations, of the sources. Unfortunately, it cannot be solved in the general case. The approach in this monograph is to replace integrals with summations. Although appealing and simplifying, it leads to two fundamental questions.

C.3.1 What is the correlation between the various frequencies of one source?

There is no issue if the source is single frequency. If the source is periodic, but not sinusoidal (e.g., a square wave), Fourier analysis shows a harmonic structure. The level at each harmonic frequency has a defined phase relationship to the fundamental and is taken into account at a distance, since the waveform is preserved.

It is common practice to collect broad-band random sound data in one-third or one-octave bands. How are the bands to be summed to get the overall level? If the sound is caused by turbulent flows, the question becomes: What is the correlation between the frequencies associated with different eddy sizes at a particular location? It is generally assumed that such bands can be added incoherently, but it may not be valid always. Incoherent addition is more likely to be justified when the bandwidth is wide. One-third octave bands are added incoherently in this monograph.

C.3.2 What is the correlation between spatially separate sources?

Much of the modeling in this monograph is associated with breaking a distributed source into a finite number of individual uncorrelated sources and then summing them to get the overall level at a particular location. The interaction between two single frequency sources can vary from complete correlation to none. In several of the examples in this monograph, the second source is an image of the first, e.g. reflection from a surface. In this case, they are completely correlated and summing independent sources fails. The case of partial correlation, such as reflection from an absorptive surface is not addressed.

The interaction of two broad-band sources is more interesting. The sound from the trailing edge of an airfoil caused by turbulent boundary layer flow is one example. The various frequency bands are created by the turbulent eddies, and as noted above are added incoherently to determine overall. But what is the relationship between two eddies that pass the trailing edge simultaneously, but are laterally separated? There has to be a distance at which where they are not aware of each other and they can be treated as incoherent (independent) sources? It is highly likely that the distance depends on eddy size and therefore is a function of frequency. Unfortunately, this information is not available for many source sources and without this knowledge, the validity of modeling by summing independent sources is weak. The approach taken here is to model a sound source with a variable number of independent sources and compare the difference that occur. If data were available, the results could be compared with experiment to determine which distance is most correct.
C.4 Correlation and Covariance Equations

Correlations are used to relate two random variables: \(x(r_1,t)\) and \(y(r_2,t)\). The mean value for each variable can be expressed as

\[
\eta_x(r_1) = \mathbb{E}\{x(r_1,t)\} \quad (C.2)
\]

\[
\eta_y(r_2) = \mathbb{E}\{y(r_2,t)\}
\]

where \(r_1\) and \(r_2\) are the positions at which the data are taken. From this some correlation terms can be defined:

**Auto Correlation**

\[
R_{xx}(r_1,t_1,t_2) = \mathbb{E}\{x(r_1,t_1) \cdot x(r_1,t_2)\} = x(r_1,t_1) \cdot x(r_1,t_1 + \tau)
\]

**Cross Correlation**

\[
R_{xy}(r_1,r_2,t_1,t_2) = \mathbb{E}\{x(r_1,t_1) \cdot y(r_2,t_2)\} = x(r_1,t_1) \cdot y(r_2,t_2 + \tau)
\]

**Auto Covariance**

\[
C_{xx}(r_1,t_1,t_2) = R_{xx}(r_1,t_1,t_2) - \eta_x(r_1) \cdot \eta_x(r_1) = R_{xx}(r_1,\tau) - \eta_x^2
\]

**Cross Covariance**

\[
C_{xy}(r_1,r_2,t_1,t_2) = R_{xy}(r_1,r_2,t_1,t_2) - \eta_x(r_1) \cdot \eta_y(r_2) = R_{xy}(r_1,r_2,\tau) - \eta_x \cdot \eta_y
\]

Note that all these equations are dimensional, typically mean square values. They can be calculated for two different positions \(r_1\) and \(r_2\), two different times, \(t_1\) and \(t_2\), or a time delay \(\tau = t_2 - t_1\). The **covariance** is the difference of the **correlation** from the mean square value. Instead of the mathematical mean values expressed in Eqs. C-2, they are replaced in the above equations by physically realizable measures. The ensemble measure is replaced by a time history (typically only one) where \(T\) is considerably less than the infinity suggested in Eq. C-4.

\[
\eta_x(r_1) = \lim_{T \to \infty} \frac{1}{2T} \int_T^{-T} x(r_1,\tau) d\tau = x(r_1,t)
\]

Other metrics used are the **correlation coefficients**. They are second order measures.

**Auto Correlation Coefficient**

\[
T_{xx}(r_1,\tau) = \frac{x(r_1,t) \cdot x(r_1,t + \tau)}{x^2(r_1,t)} = \frac{R_{xx}(r_1,\tau)}{R_{xx}(r_1,0)} \quad (C.5)
\]

**Cross Correlation Coefficient**

\[
T_{xy}(r_1,r_2,\tau) = \frac{x(r_1,t) \cdot y(r_2,t + \tau)}{\sqrt{x^2(r_1,t) \cdot y^2(r_2,t)}} = \frac{R_{xy}(r_1,r_2,\tau)}{\sqrt{R_{xx}(r_1,0) \cdot R_{yy}(r_2,0)}}
\]

It can be shown that the range of the coefficients is from −1 to +1, minus related to negative correlations and positive related to positive correlations (Figure C-1). Higher moments can also be defined. Note that these equations are dimensionless, so are independent of the measurement system used and are more useful in learning about sound sources.

In acoustics, the measurement system rejects D.C. values so the means are zero; there is little difference between covariance and correlation. Although **covariance** is the more correct term, use of the more common word **correlation** should not introduce misinterpretations.
Note that all the above equations are based on the *Ergodic hypothesis*; a measure along a time history is equivalent to a measure across a large number of samples at one time.

### C.5 Spectral Density

Spectral density is a measure of how much of a given variable resides in a band of frequencies 1 Hz wide. It is a second moment property of the process, so is related to the product of variables. If these products are related to variables such as the square of pressure or velocity, they may be related to power, in which case they may be referred to as *power spectral density*. It is common practice to use the prefix “power” for any spectral density, while here is it properly used only for acoustical power. The other power related variable is the *intensity spectral density* that can be measured directly with proper instruments, or deduced from sound pressure in the far sound field.

As an example, the cross spectral densities are related to the correlations by the transform pair.

\[
S_{xy}(r_1, r_2, \omega) = \int_{-\infty}^{\infty} R_{xy}(r_1, r_2, \tau) e^{-i\omega \tau} d\tau
\]

\[
R_{xy}(r_1, r_2, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(r_1, r_2, \omega) e^{i\omega \tau} d\omega
\]

Note that these variables are dimensional. The dummy variable \( \tau \) can be either positive or negative. Note also that there is no phase information between the various frequencies for either formulation so information is lost, thus it is not possible to reconstruct the record from which the data were taken. In any real system, negative frequencies are indistinguishable from positive frequencies, so the integrals are reduced to one-sided ones, in which the negative frequency domain is folded over.

The mean square value and the correlation can be given in terms of the spectral density.

\[
x^2(r_1, t) = \frac{1}{\pi} \int_{0}^{\infty} S_{xx}(r_1, \omega) d\omega
\]

\[
R_{xx}(r_1, r_2, \tau) = \frac{1}{\pi} \int_{0}^{\infty} S_{xx}(r_1, r_2, \omega) \cos \omega \tau d\omega
\]

The spectral density of a variable, such as source strength, is

\[
S_{00}(\omega) = \lim_{T \to \infty} \frac{1}{2T} \left[ Q(\omega) Q^*(\omega) \right]
\]

The above equations can make mathematicians happy. The results are useful to engineers, but are difficult to obtain without a research project. The approach here, when using one-third octave band analysis, is to replace the integrals with summations and only over a finite frequency range. In the *SoundSource* program, each one-third octave band is presumed to be composed of white noise prior to band summations; \{C.6.2\} discusses converting one-third octave band levels to spectrum levels based on this approximation.
One-third octave band data are easy to obtain and details of these bands are discussed in \{C.6.1\}.

**C.6 Measurement Bandwidths**

The need for integration in making approximate analyses of sound sources can be bypassed by using one-third octave band spectra and assume that each band is composed of band limited white noise. Band spectra can be easily measured or modeled based on similar sources. The next section \{C.6.1\} shows the lower, center, and upper band limits for both one and one-third octave bands in common use. Section \{C.6.2\} permits a one-third octave band to be reduced to spectrum level based on it being composed of band limited white noise. It then becomes possible to computer model the directivity characteristics of a particular source type. For example, it is possible to obtain the directivity characteristics of a broad-band dipole source above a rigid plane surface at an arbitrary orientation to the surface.

**C.6.1 Band Frequency Limits**

Typical measurement systems use either one octave or one-third octave bands, while most analysis is on a spectral basis. \(B\) is the half-width of three common band widths, and \(\omega_c\) is the center frequency of the band.

The equations to locate the lower, center, and upper frequencies for the standard ANSI bands are shown in Eqs. C.8.

\[
\begin{align*}
CFreq(BN) &= 10^{BN/10} \\
LFreq(BN) &= CFreq(BN) \times 2^{-1/M} \\
UFreq(BN) &= CFreq(BN) \times 2^{1/M}
\end{align*}
\]

This is equation (C.8)

characteristics of a particular source type. For example, it is possible to obtain the directivity characteristics of a broad-band dipole source above a rigid plane surface at an arbitrary orientation to the surface.

Typical measurement systems use either one octave or one-third octave bands, while most analysis is on a spectral basis. \(B\) is the half-width of three common band widths, and \(\omega_c\) is the center frequency of the band.

The equations to locate the lower, center, and upper frequencies for the standard ANSI bands are shown in Eqs. C.8.

\(BN\) stands for the ANSI band number \((30=1000 \text{ Hz})\). \(M\) is the band divisor: 2 for octave bands, 6 for one-third octave bands. Table C-2 shows the results for commonly used frequency bands. The frequencies in parentheses are the commonly used descriptors for each band. Note that the center frequencies are based on powers of ten \([49, 50]\) while bandwidths are based on powers of two. In realizable filters, some band overlaps occur, primarily at the higher frequencies, but they are relatively minor. If the level in one band is much higher than that in the contiguous bands, measurement errors can occur.
For calculating the interaction of two sources with only one-third octave band spectra, a simple approximation is to assume that each band is composed of band-limited white noise. In that way, each frequency within the band can be analyzed separately. Consider the value of $k_r$ for a one-third octave band centered on 1000 Hz. There is a 25% difference in $k_r$ within this one band, so significant level differences can occur. For this approximation, the relationship is

$$L_s = L_p(Band) - 10 \log_{10}(BW)$$
L_s is the white noise spectrum level in the band with the band level, L_p and BW is the bandwidth (the difference between the upper band limit and the lower band limit) in the particular band. Table C-3 shows the numerical corrections for one-third octave bands.

<table>
<thead>
<tr>
<th>ANSI Band Number</th>
<th>Nominal One-Third Octave Band Center Frequency</th>
<th>Correction from Band Level to Spectrum Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>31.5</td>
<td>-8.6</td>
</tr>
<tr>
<td>16</td>
<td>40</td>
<td>-9.6</td>
</tr>
<tr>
<td>17</td>
<td>50</td>
<td>-10.6</td>
</tr>
<tr>
<td>18</td>
<td>62.5</td>
<td>-11.6</td>
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<tr>
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<td>80</td>
<td>-12.6</td>
</tr>
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<td>20</td>
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<td>-16.6</td>
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<tr>
<td>24</td>
<td>250</td>
<td>-17.6</td>
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<td>400</td>
<td>-19.6</td>
</tr>
<tr>
<td>27</td>
<td>500</td>
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</tr>
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<td>28</td>
<td>630</td>
<td>-21.6</td>
</tr>
<tr>
<td>29</td>
<td>800</td>
<td>-22.6</td>
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<tr>
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<td>35</td>
<td>3150</td>
<td>-28.6</td>
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<tr>
<td>36</td>
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<td>37</td>
<td>5000</td>
<td>-30.6</td>
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<tr>
<td>38</td>
<td>6300</td>
<td>-31.6</td>
</tr>
<tr>
<td>39</td>
<td>8000</td>
<td>-32.6</td>
</tr>
<tr>
<td>40</td>
<td>10000</td>
<td>-33.6</td>
</tr>
</tbody>
</table>

Table C-3. Correction from one-third octave band level to spectrum level, based on white noise.

**C.7 Band Limited White Noise**

Modeling of broad-band sound in this monograph relies on conversion of bands to spectrum levels based on the white noise approximation. Band limited white noise, of band width b, can be expressed in the form

\[ S_{xx}(r, \omega) = S \quad \omega_{c} - b/2 \leq \omega \leq \omega_{c} + b/2 \]

\[ S_{xx}(r, \omega) = 0 \quad \text{elsewhere} \]

It is instructive to look at the structure of this noise. The auto-correlation for it is

\[ R_{xx}(r, \tau) = \frac{1}{\pi} \int_0^\infty S_{xx}(r, \omega) \cos \omega \tau d\omega = \frac{bS}{\pi} \int_{\omega_{c} - b/2}^{\omega_{c} + b/2} \cos \omega \tau d\omega = \frac{bS}{\pi} \frac{b\tau}{2} \cos \omega_{c} \tau \]

C-8
The auto-correlation coefficient simply divides the correlation by its value at $\tau=0$, so the multiplier in the above equation becomes one. Figure C-2 shows this coefficient for a one-third octave band centered at 100 Hz. The sine function acts as a damped envelope around the center frequency. The coefficient oscillates about zero and decays to near zero in about five periods of the center frequency. Note that as the bandwidth decreases toward zero, the sine term approaches one, suggesting that the correlation coefficient extends toward infinity as would be expected for a sine wave.

![One-Third Octave Band White Noise Centered at 100 Hz](image)

**Fig. C-2.** The auto-correlation coefficient at 100 Hz for a one-third octave band.

The auto-correlation coefficient is a measure of the relationship of a signal at one time with that at an earlier time. Similar relationships occur spatially for turbulent flows. Figure C-3 shows for a white noise spectrum centered at 100 Hz for two bandwidths (one and one-third octave). The broader the bandwidth, the more quickly the coefficient decays. Figure C-4 shows the auto-correlation coefficient for a white noise spectrum centered at 1000 Hz for two bandwidths (one and one-third octave). The coefficient decays more quickly with broader bandwidths. The higher the frequency the more rapidly the coefficient decays. Figure C-5 shows the time for band-limited white noise to lose correlation (a logarithmic decay with frequency).

**Key Point:** Most of the fluid mechanical models in this monograph for broad-band sound are based on an arbitrary, but variable, number of incoherent sources e.g., {4.9.5}. The results above suggest that smaller turbulent eddies are associated with higher frequencies and smaller correlation distances. To properly model a broad band sound source, say trailing edge noise {4.9.2.2}, each frequency band must have a different number of sources laterally. Since the primary emphasis was on defining the source type and characteristic variables this was considered an unnecessary refinement.
Fig. C-3. The correlation coefficient at 100 Hz for two bandwidths.

Fig. C-4. The auto-correlation coefficient at 1000 Hz for two bandwidths.

Fig. C-5. The correlation decays rapidly for higher frequencies and broader bandwidths.
Appendix D
Development of the Wave Equation

D.1 The Equations of Fluid Motion

The equations of fluid motion are embodied in the equations of mass, momentum and thermodynamics. They will be expressed in Cartesian tensor notation, as it is more compact. See Appendix B for notation details. The development of these equations is attributed to Claude-Louis Navier (1785-1836) and Sir George Gabriel Stokes (1819-1903) and they are often called the Navier-Stokes equations. Although the developers were concerned primarily with incompressible flows, the equations are sufficiently general to handle compressible flows. These equations express the conditions in an infinitesimal cube of space by calculating the in/out flow and the changes of mass and momentum inside.

D.1.1 The Mass Continuity Equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = \nabla \cdot (\rho u) = 0 \tag{D.1}
\]

where \(\rho\) is the fluid density, \(t\) the time, \(x_i\) represents the spatial coordinates and \(u_i\) the respective fluid velocities. The subscript \(i\) varies from 1 to 3 (no relation to \(\sqrt{-1}\)). The second term is the net in/out flow of fluid which results in a change of internal density. If zero, the density must be constant, so no sound generation occurs \{1.2.1\}.

D.1.2 The Momentum Equation

Approximation 1. The fluid is Newtonian. The general form for a normal (Newtonian) fluid is

\[
\frac{\partial}{\partial t} (\rho u_j) + \frac{\partial}{\partial x_i} (\rho u_i u_j) = \frac{\partial P_i}{\partial x_j} + \varepsilon E_i - \frac{\partial H_{ij}}{\partial x_j} \tag{D.2}
\]

A Newtonian fluid, named for Isaac Newton (1643–1727), is a fluid where the stress versus rate of strain curve is linear and passes through the origin. The constant of proportionality is called viscosity. Air and water are Newtonian fluids. The fluid response on the left is responding to the surface forces \(P_{ij} = p\delta_{ij} - \sigma_{ij}\). Where \(p\) is the static pressure, \(\delta_{ij}\) is the Kronecker delta function and \(\sigma_{ij}\) is the viscous stress tensor. The fluid also responds to an electric field, where \(\varepsilon\) is the charge density and \(E\) the electric field vector. The fluid responds also to the magnetic field where \(H_{ij}\) is the magnetic stress tensor.

Approximation 2. The electromagnetic forces are of no interest, so the momentum equation reduces to

\[
\frac{\partial}{\partial t} (\rho u_j) + \frac{\partial}{\partial x_i} (\rho u_i u_j) = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial u_i}{\partial x_j} \tag{D.2}
\]

(D.1)
The viscous term on the right includes the dynamic viscosity. Although spherical coordinates are used in this monograph, the complete momentum equation in that coordinate system is excessively long and not illuminating.

**D.2 Dynamic Similarity**

When studying engineering problems, dynamic similarity is often neglected. Nature knows nothing of the dimensional systems we use, but knows only the ratio of quantities, such as lengths and forces. Important information can be derived by putting the equations of motion in dimensionless form and from that dynamic similarity ratios can be derived. This is done by relating the physical variables to reference quantities such as shown in the list below. The values subscripted with a 1 are now dimensionless. By convention, the dimensionless density is given the symbol \( s \). Two of those reference quantities \((L, U)\) can be called *characteristic numbers* and constitute what may be called *scaling rules*. There is no loss in generality if Eqs. D.1 and D.2 are reduced to one-dimensional Cartesian coordinates where only motion along the \( x \) axis occurs. These equations are shown in Eqs. D.3.

\[
\begin{align*}
\rho &= \rho_0 (1 + s) \\
p &= p_0 (1 + p_1) \\
u &= U u_i \\
x &= L x_i \\
t &= \omega t \\
c_0^2 &= \frac{\gamma p_0}{\rho_0}
\end{align*}
\]

Substituting the new variables into Eqs. D.3, the equations are now expressed in dimensionless form.

\[
\begin{align*}
2\pi St \frac{\partial}{\partial t} \left[ (1 + s) u_1 \right] + \frac{\partial}{\partial x_1} \left[ \left(1 + s \right) u_1 \right] &= 0 \\
2\pi St \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i} &= -\frac{1}{\gamma Ma^2 (1 + s)} \frac{\partial p_1}{\partial x_i} + \frac{1}{Re (1 + s)} \frac{\partial^2 u_i}{\partial x_i^2}
\end{align*}
\]

The kinematic viscosity \( \nu \) has been introduced. Three important dimensionless parameters are now obvious; see \{A.2.2\} for more details on them.

\[
\begin{align*}
Re &= \frac{UL}{\nu} & \textbf{Reynolds number}. & \text{The ratio of the steady inertial forces to the steady viscous forces.} \\
St &= \frac{fL}{U} & \textbf{Strouhal number}. & \text{The ratio of unsteady inertial forces to steady inertial forces.} \\
Ma &= \frac{U}{c_o} & \textbf{Mach number}. & \text{The ratio of the steady speed to the speed of sound.}
\end{align*}
\]
These numbers are extremely useful in understanding the nature of sound sources and may be considered essential for evaluating dynamic similarity. If the characteristic length $L$ and the characteristic speed $U$ are chosen properly, much can be learned about the physics of a particular problem. Inversely, information obtained from a sound source can be used to get good estimates of what these two characteristic variables might be. As is shown in the chapters, these variables are best thought of as local ones, e.g., the steady speed of a jet varies with distance from the nozzle.

### D.3 The Equation of State

In general, the equation of state for a gas can be expressed in the form

$$ p = p(\rho, e) $$

$$ dp = \frac{\partial p}{\partial \rho} d\rho + \frac{\partial p}{\partial e} de $$

Where $p$ is the pressure, $\rho$ is the density, and $e$ is the entropy.

**Approximation 3.** The motion is isentropic (adiabatic and reversible) so we can define a sound speed.

**Approximation 4.** It is a perfect gas: $p=\rho RT$ and $p=K\rho^\gamma$

$$ c^2 = \frac{\partial p}{\partial \rho} |_{e} $$

$$ c^2 = \frac{\gamma P}{\rho} = \gamma RT $$

$$ c^2 = \frac{\gamma \rho p_0(1+p_1)}{\rho_0(1+s)} $$

$$ p_1 = \frac{p - p_0}{p} $$

$$ s = \frac{\rho - \rho_0}{\rho_0} $$

$T$ is the absolute temperature, $K$ is a constant and $\gamma$ is the ratio of specific heats.

A perfect gas is one in which the molecules have elastic collisions. It is a good approximation to the behavior of air under many conditions. Émile Clapeyron (1799-1864), August Kronig (1822-1879), and Rudolf Clausius (1822-1888) helped to develop the relevant equations.

**Approximation 5.** The magnitude of the pressure fluctuations $p_1$ and those of the density $s$ are so small relative to one that they can be neglected. The sound speed is now

$$ c_0^2 = \frac{\gamma p_0}{\rho_0} = \gamma RT_0 $$

(D.6)

D-3
A typical value of $\gamma$ is 1.4 and that for $R$ is 53.3 ft-lb/lb$^\circ$R. Note that all of the above approximations require that the sound field be a small perturbation on the static condition. This is not always true \{2.2.6\}. Note also that in order to have a propagation speed, and therefore sound, the medium must be elastic (the density must vary). Since both solids and fluids are elastic, most (but not all) time varying motions will generate sound. There are many cases where the sound created is unheard because it is either too low in level or outside the frequency range of listeners \{1.6\}.

**D.4 Simplifying to the Wave Equation**

The sound field is a subset of the fluid motion, so the acoustic wave equation must be imbedded somewhere in the equations of fluid motion. How deeply is it buried? In the general case, cross differentiating the continuity and momentum equations (Eqs. D.1 and D.2) yields

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial t} \left( \rho u_i \right) \right) = 0$$

(D.7)

$$\frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial t} \left( \rho u_i \right) \right) + \frac{\partial^2}{\partial x_i \partial x_j} \left( \rho u_i u_j \right) = \frac{\partial^2 p_i}{\partial x_i \partial x_j}$$

The development of the acoustic wave equation from these equations is given in a number of texts. A particularly illuminating approach was done by Sir Michael Lighthill (1924-1998) who developed the theory of sound generated aerodynamically \[12,13\]. For completeness, his form of the wave equation is presented below, but it is still too general.

$$\frac{\partial^2 \rho}{\partial x_i^2} - \frac{1}{c_o^2} \frac{\partial^2 \rho}{\partial t^2} = -\frac{1}{c_o^2} \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

(D.8)

$$T_{ij} = \rho u_i u_j + p_i - c_o^2 \rho \delta_{ij}$$

Rather than repeat the general development and with no loss of generality, the wave equation for the one dimensional case is developed. Eqs. D.4 suggest approximations that can be made.

**Approximations 3, 4, 5.** See previous pages.

**Approximation 6.** The Reynolds number is large enough so that viscous terms can be neglected. See \{2.3.7\}.

**Approximation 7.** The Mach number is small enough so that the pressure gradient term can be retained.

**Approximation 8.** The third term in the continuity equation is much smaller than the second.

**Approximation 9.** The second term in the momentum equation is much smaller than the first.

With these approximations, Eqs. D.3 are reduced to the following. The terms in red are deleted.

$$\frac{\partial s + \partial u}{\partial t} + \frac{\partial s u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho_o} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} = \frac{1}{\rho_o} \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial x} = c_o^2 \frac{\partial s}{\partial x}$$

(D.9)
Cross differentiating the two equations yields the wave equation for density fluctuations.

\[
\frac{\partial^2 s}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 s}{\partial t^2} = 0
\]  

(D.10)

The condensation is related to the other physical variables \{A.3\} so the wave equation can be made to apply to each of them.

**D.5 Validity of Approximations**

It is necessary to show \textit{a posteriori} under what conditions the approximations are valid.

**Approximation 1.** The discussion in this monograph applies only to air so that this approximation is applicable. Slight deviations from a Newtonian fluid may not create problems, since the influence of viscosity is sufficiently small to be neglected.

**Approximation 2.** The discussion in this monograph does not apply to extensive electromagnetic fields. If the effect is actual but contained \textit{within} the hypothetical surface, it has no influence on the wave equation.

**Approximation 3.** If the motion is small enough and quick enough, the motion will be adiabatic and reversible. An abstract from Eqs D.5 is given on the right. From Table 4-1, at a level of near 88 dB, \( p_1 = 4.72 \times 10^{-6} \) and \( s = 3.43 \times 10^{-6} \), more than sufficient to meet the approximation.

**Approximation 4.** The discussion in this monograph is restricted to air in the volume outside the hypothetical surface, so the perfect gas approximation is good.

**Approximation 5.** The magnitude of the pressure and density are sufficiently small to neglect nonlinear terms. See Approximation 3.

**Approximation 6.** The Reynolds number of the steady flow in most situations is sufficiently large so that viscous effects can be neglected. See \{2.3.7\}.

**Approximation 7.** The Mach number, if subsonic, defines the boundary between essentially incompressible flow and compressible flow (regardless of the sound field). \textit{If the hypothetical boundary} is at an appropriate distance (outside any flow field), the \textit{local} Mach number is sufficiently small to make the approximation valid.

**Approximation 8.** The inequality \(|su| \ll |u|\) is easily met if Approximation 3 is met.

**Approximation 9.** The inequality \(|u \frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial t}|\) is met if \( M \ll 1 \). Although the characteristic speed was not restricted in forming Eqs. D.4, the speed for this case is that of the so-called \textit{acoustical particle} motion, which implies a restriction that the medium be at rest.

The wave equation is normally satisfied for normal sound levels experienced outside the hypothetical surface. Next, it is worthwhile to consider where that surface should be by examining the various fields that surround a source.
### D.6 Regions Around a Sound Source

It is instructive to look at the influence of the various approximations as the distance from a small sound source is decreased: the monopole model is used as an example. Far from the source, the wave equation is valid, but as the source is approached, the approximations needed to make that equation valid are violated, so the equations used for analysis must change. Those changes are listed below and are shown graphically in Figure D.1. It is important to note that the equations of motion (Eqs. D.1) always apply everywhere outside a surface within which chemical or nuclear reactions occur. The relationships shown below are just those used to analyze the dominant features of the fluid motion.

It is presumed that the medium, through which the sound passes, is at rest.

#### D.6.1 The Sound Field Region

The momentum convection term can be neglected as noted in Eqs. D.10. The local Strouhal number is useful and is based on the distance and local particle speed. The local Mach number is also defined the same way and the subscript l is applied to denote the difference.

Using Eqs. 3.7, the dependence on distance of the two local parameters and the condensation are shown on the right. The influence of the density change term in the continuity equation becomes more prominent as the field becomes more “acoustical” with greater distance and decreases with reduction of distance. The requirement for the condensation to be small is shown in the relationship on the right; there is an upper limit on both frequency and source strength for the wave equation to be valid. When it is, the relevant equations are

\[
\frac{\partial u}{\partial t} = - \frac{1}{\rho_0} \nabla p \\
p = \rho_0 \frac{\partial \phi}{\partial t} \\
\Box^2 \phi = 0
\]

The pressure gradient has been generalized to avoid geometric complexity.

#### D.6.1.1 Geometric Far Field. \(r >> h\).

Considering the interaction of two point sound sources in Chapter 3, the separation distance between them was given as \(2h\). An approximation where \(kr_i \approx k(r - h \cos \theta)\) was made to simplify the calculations (Eqs. 3.18). Two aspects of that approximation are important: relative magnitude and relative phase. It is common to assume that the distance factor for each source is the same. If the direction is along the line between the sources (worst case) and \(r = 10h\), the magnitudes of the two inverse distances are 20% different. The phase differences created by the above approximation are only important at high frequencies. It is important for the measurement point to be as large a multiple of the distance between the sources as possible.
D.6.1.2 Sound Far Field. \( kr \gg 1 \).

The ratio of the distance to the sound wavelength is much greater than one. Examining the data in Chapter 3, the reactive component of the intensity vector is 10\% of the radial component at \( kr=10 \). The pressure and velocity are essentially in phase. This has importance for deducing intensity from sound pressure measurements. The actual distance to meet this recommendation is a function of frequency. The distance may be chosen based on the equation on the right; the second term is for normal temperatures. Table D.1 below provides some numerical results.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Frequency, Hz} & 50 & 100 & 500 & 1000 \\
\hline
kr=10 & 36 Feet & 18 Feet & 43 Inches & 22 Inches \\
\hline
\end{array}
\]

*Table D.1. The distance to the far sound field depends on frequency.*

D.6.1.3 Sound Intermediate Field. \( kr \approx 1 \).

The ratio of the distance to the sound wavelength is near one. Both reactive and resistive terms must be included in the equations. The pressure and velocity are approximately at 45 degrees, suggesting the balance between the compressible and incompressible aspects of the motion.

D.6.1.4 Sound Near Field. \( kr << 1 \).

The ratio of the distance to the sound wavelength is much less than one. The reactive components are significant and the resistive component (the sound field) is less significant. The pressure and velocity are nearly at 90 degrees to each other, suggesting the dominance of the incompressible aspect of the flow. The sound field is buried inside the incompressible flow field. If \( kr=0.1 \), the sound is only 10\% of the incompressible motion.

D.6.2 The Hydrodynamic (Quasi-incompressible) Flow Region

The incompressible aspects of the flow are dominant in this region. The momentum of the flow is more significant. The validity of the wave equation is now in doubt. The buried sound field is sufficiently small that it is often neglected in favor of the presumption that the density is constant as shown in the equations on the right.

D.6.2.1 Hydrodynamic Far Field. \( kr << 1 \).

This field overlaps the near sound field, so either approach, the compressible or incompressible, can be used. Most of the pressure is used to accelerate the fluid. If the sound speed is set to infinity, both the pressure and velocity, which are independent of the sound speed, are unaffected, the condensation becomes zero, suggestive of incompressible flow.

D.6.2.2 Hydrodynamic Intermediate Field.

The flow is sufficiently incompressible that the density change has little influence on the flow field. The momentum of the flow is sufficient that the wave equation is an invalid description of the flow. Although it can be neglected in any calculations, the sound field still exists.
D.6.2.3 Hydrodynamic Near Field.

The incompressible components totally dominate the motion so it is reasonable to treat the flow as completely incompressible. Two relevant equations are shown on the right.

\[
(u \cdot \nabla)u = -\frac{1}{\rho_0} \nabla p
\]

\[
p + \frac{1}{2} \rho_0 u^2 = \text{Const}
\]

D.6.3 The Compressible Flow Region

Closer to the source the density fluctuation becomes so large it can no longer be treated as a constant. This region is generally treated as high speed compressible flow. Two relevant equations are shown on the right.

\[
(u \cdot \nabla)u = -\frac{1}{\rho} \nabla p
\]

\[
\frac{\gamma - 1}{\gamma \rho} p + \frac{u^2}{2} = \text{Const}
\]

D.6.3.1 Subsonic Field. \( M<1 \)

At high subsonic Mach numbers, the equations of motion still allow for sound to propagate to other parts of the medium. For this case the acoustic particle motion is no longer small and is better interpreted as an oscillatory flow speed in calculating the Mach number. The density change is sufficiently large that the speed of sound equation is no longer valid within this region. This occurs with large amplitude motions where the wave shape is progressively modified toward a shock wave \{2.2.6\}. Low power explosions are examples of subsonic transients of high amplitude.

D.6.3.2 Supersonic Field. \( M>1 \)

At supersonic Mach numbers, the equations of motion do not allow sound to propagate to other parts of the medium. Shock waves are common expressions of this motion. High power explosions or nuclear shock waves are examples.

D.6.4 The Reaction Region

Within this region, either chemical or nuclear reactions occur. The usual Navier-Stokes equations are not applicable. Depending on the nature of the reactions this region can replace, or overlap, the compressible and incompressible regions. At one extreme is a propane torch where its source magnitude is small enough so that the hypothetical surface can be reasonably close. At the other extreme is the H-bomb, the hypothetical surface is likely to be incinerated at any reasonable distance.

D.7 Key Points

Looking at the details of a sound source shows the potential complexity of mathematically approaching the source. When the source is treated as a black box with a hypothetical surrounding surface outside of which the wave equation for a still medium is valid, much useful information can be gained. The challenge is to choose the correct distance of that surface from the source. The discussion above shows that this can be a difficult choice and the intent here is make the reader aware of what items must be considered to make that choice. If any measurements are made within that bounding surface, substantial errors of level and spectrum can occur. No attempt has been made to look at a medium in motion. That adds (or subtracts) a steady speed to the sound speed, with resulting time delays, level, and Doppler...
shifts. Most ordinary sound problems occur at low enough Mach numbers where these effects are not significant.

<table>
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<tr>
<td>$Ma&gt;1$</td>
<td>Supersonic Flow</td>
</tr>
</tbody>
</table>

Chemical and Nuclear

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**Fig. D-1. Regions of fluid motion around a sound source.**
References

16. Sonic Systems, “SoundSphere Speakers”
49. Anon., “Preferred Frequencies, Frequency Levels, and Band Numbers for Acoustical Measurements”, ANSI S1.6-1984